
Problem Sheet 9 [1]

Based on Chapters 7.1 - 7.4 of the course book.

Optional Revision Problems

Exercise 1. Let $X \sim Expo(\lambda)$. Find the median and mode of X.

Exercise 2. Let $X \sim Bin(n, p)$.

1. For $n = 5$, $p = 1/3$, find all medians and all modes of X. How do they compare to the mean?

2. For $n = 6$, $p = 1/3$, find all medians and all modes of X. How do they compare to the mean?

Week 9 Exercises

Exercise 3. Alice, Bob, and Carl arrange to meet for lunch on a certain day. They arrive independently at uniformly distributed times between 1 pm and 1:30 pm on that day.

1. What is the probability that Carl arrives first?

   For the rest of this problem, assume that Carl arrives first at 1:10 pm, and condition on this fact.

2. What is the probability that Carl will be waiting alone for more than 10 minutes?

3. What is the probability that Carl will have to wait more than 10 minutes until his party is complete?

4. What is the probability that the person who arrives second will have to wait more than 5 minutes for the third person to show up?

Exercise 4. A fair coin is flipped twice. Let X be the number of Heads in the two tosses, and Y be the indicator r.v for the tosses landing the same way.

1. Find the joint PMF of X and Y.

2. Find the marginal PMFs of X and Y.

3. Are X and Y independent?

4. Find the conditional PMFs of Y given $X = x$ and of X given $Y = y$.

[1] Exercises are based on the coursebook Statistics 110: Probability by Joe Blitzstein

Exercise 5. Let X and Y have joint PDF $f_{X,Y}(x, y) = cxy$, for $0 < x < y < 1$.

1. Find c to make this a valid joint PDF.
2. Find the marginal PDFs of X and Y.
3. Are X and Y independent?
4. Find the conditional PDF of Y given $X = x$.

Exercise 6. Two students, A and B, are working independently on homework assignments. Student A takes $Y_1 \sim Expo(\lambda_1)$ hours to finish their homework, while B takes $Y_2 \sim Expo(\lambda_2)$ hours.

1. Find the CDF and PDF of Y_1/Y_2, the ratio of their problem-solving times.
2. Find the probability that A finishes their homework before B does.

Exercise 7. Let X and Y be discrete r.v.s.

1. Use 2D LOTUS (without assuming linearity) to show that $E(X + Y) = E(X) + E(Y)$.

   Hint: If you are really stuck, check Example 7.2.4, but after that, this exercise reduces to copying.

2. Now suppose that X and Y are independent. Use 2D LOTUS to show that $E(XY) = E(X)E(Y)$.

Correlation Exercises

If during the lecture the definition of *covariance* and *correlation* are not covered, leave these exercises for next week.

Exercise 8. Two fair, six-sided dice are rolled (one green and one orange), with outcomes X and Y for the green die and the orange die, respectively.

1. Compute the covariance of $X + Y$ and $X - Y$.
2. Are $X + Y$ and $X - Y$ independent?

Exercise 9.
1. Let X and Y be Bernoulli r.v.s, possibly with different parameters. Show that if X and Y are uncorrelated, then they are independent.
2. Give an example of three Bernoulli r.v.s such that each pair of them is uncorrelated, yet the three r.v.s are dependent.

   Hint: Start with two independent Bernoulli r.v.s and define the third one cleverly as a function of the first two.