Problem Sheet 8 [1]

Based on Chapters 5.4, 5.5, part of 5.7, and Chapter 6.1 of the course book.

Optional Revision Problems

Exercise 1. The Rayleigh distribution has PDF $f(x) = xe^{-x^2/2}$, $x > 0$, and zero otherwise. Let X have the Rayleigh distribution.

1. Find $P(1 < X < 3)$.
2. Find the first quartile, median, and third quartile of X; these are defined to be the values q_1,q_2,q_3 (respectively) such that $P(X \leq q_j) = j/4$ for $j = 1, 2, 3$.

Hint: For calculating probabilities from the PDF, you can use the information in the table in Section 5.8 in the book.

Exercise 2. A stick is broken into two pieces, at a uniformly random breakpoint. Find the CDF and average of the length of the longer piece.

Week 8 Exercises

Exercise 3. Let $U \sim Unif(0, 1)$. Using U, construct $X \sim Expo(\lambda)$

Exercise 4. Let $Z \sim \mathcal{N}(0, 1)$. Create an r.v. $Y \sim \mathcal{N}(1, 4)$, as a simple-looking function of Z. Make sure to check that your Y has the correct mean and variance.

Hint: If the transformation is not immediate, check Definition 5.4.3

Exercise 5. Let $Z \sim N(0, 1)$. We know from the 68-95-99.7% rule that there is a 68% chance of Z being in the interval $(-1, 1)$. Give a visual explanation of whether or not there is an interval (a, b) that is shorter than the interval $(-1, 1)$, yet which has at least as large a chance as $(-1, 1)$ of containing Z.

Exercise 6. A post office has 2 clerks. Alice enters the post office while 2 other customers, Bob and Claire, are being served by the 2 clerks. She is next in line. Assume that the time a clerk spends serving a customer has an $Expo(\lambda)$ distribution.

1. What is the probability that Alice is the last of the 3 customers to be done being served?

   Hint: No integrals are needed.

[1] Exercises are based on the coursebook Statistics 110: Probability by Joe Blitzstein

2. What is the expected total time that Alice needs to spend at the post office?

   Hint: Example 5.6.3 can be useful.

Exercise 7. Let T be the time until a radioactive particle decays, and suppose (as is often done in physics and chemistry) that $T \sim Expo(\lambda)$.

1. The half-life of the particle is the time at which there is a 50% chance that the particle has decayed (in statistical terminology, this is the median of the distribution of T). Find the half-life of the particle.

2. Show that for ϵ a small, positive constant, the probability that the particle decays in the time interval $[t, t+\epsilon]$, given that it has survived until time t, does not depend on t and is approximately proportional to ϵ.

   Hint: $e^x \approx 1 + x$ if $x \approx 0$.

Exercise 8. Let $U \sim Unif(a, b)$. Find the median and mode of U.

Exercise 9. Let Y be Log-Normal with parameters μ and σ^2. So $Y = e^X$ with $X \sim N(\mu, \sigma^2)$. Three students are discussing the median and the mode of Y . Evaluate and explain whether or not each of the following arguments is correct.

(a) Student A: The median of Y is e^μ because the median of X is μ and the exponential function is continuous and strictly increasing, so the event $Y \leq e^\mu$ is the same as the event $X \leq \mu$.

(b) Student B: The mode of Y is e^μ because the mode of X is μ, which corresponds to e^μ for Y since $Y = e^X$.

(c) Student C: The mode of Y is μ because the mode of X is μ and the exponential function is continuous and strictly increasing, so maximizing the PDF of X is equivalent to maximizing the PDF of $Y = e^X$.