143

$$b_{\mu,\sigma^2}\left(a\tilde{\sigma}_n^2\right) = \sigma^2\left(a\left(\frac{n-1}{n}\right) - 1\right)$$

$$\mathrm{Var}_{\mu,\sigma^2}\left(a\tilde{\sigma}_n^2\right) = \mathrm{Var}\left(\frac{a n\tilde{\sigma}_n^2}{n}\right) = \frac{a^2}{n^2}\mathrm{Var}\left(n\tilde{\sigma}_n^2\right) = \frac{a^2}{n^2}2\sigma^4(n-1)$$

$$\mathrm{EQM}_{\mu,\sigma^2}\left(a\tilde{\sigma}_n^2\right) = \frac{a^2}{n^2}2\sigma^4(n-1) + \sigma^4\left(a\frac{n-1}{n} - 1\right)^2$$

Il faut minimiser par rapport à a

$$\frac{\partial}{\partial a} = \sigma^4 \frac{2(n-1)}{n^2} a$$

$$+ \sigma^4 2 \left(a \frac{n-1}{n} - 1 \right) \frac{n-1}{n}.$$

$$\frac{\partial}{\partial a} = 0 \iff \boxed{a = \frac{n}{n+1}}$$

(algèbre)

$$\frac{\partial^2}{\partial^2 a} > 0 \Rightarrow \text{minimum}$$

Le meilleur estimateur est

$$\frac{1}{n+1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

pour minimiser le biais $\to a = \frac{n}{n-1}$

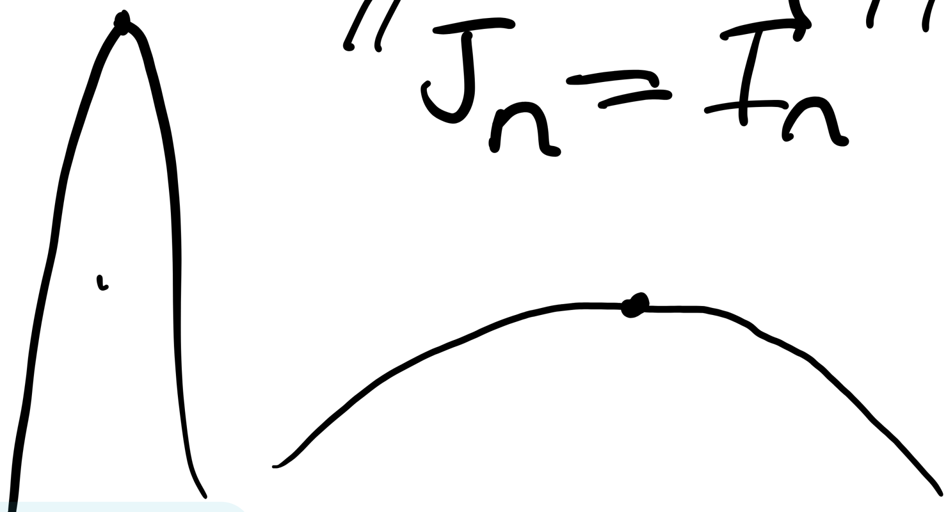
pour minimiser la var $\to a = 0$

---

166 méthode delta

avec $g = (\ell')^{-1}$

"$J_n = I_n$"



(16f) $\ell(\lambda) = n \log \lambda - \lambda n \overline{X}_n$

$$\ell'(\lambda) = \frac{n}{\lambda} - n\overline{X}_n$$

$$\ell''(\lambda) = -\frac{n}{\lambda^2}$$

$$I_n = -\mathbb{E}_\lambda\left[-\frac{n}{\lambda^2}\right] = \frac{n}{\lambda^2}$$

rien d'aléatoire

$$\hat{\lambda}_{ML} \overset{app}{\sim} \mathcal{N}\left(\lambda, \frac{\lambda^2}{n}\right)$$

(on a vu $\hat{\lambda}_{ML} = \frac{1}{\overline{X}_n}$)

$$J_n = -\ell''(\hat{\mu}_{ML}) = \frac{n}{\hat{\mu}_{ML}^2}$$

$$= n\overline{X}_n^{\,2}$$

---

171 $\Pr\left(z_{\frac{\alpha}{2}} \le Z_n \le z_{1-\frac{\alpha}{2}}\right)$

$= 1-\alpha$, où z_β est tel

que $\Pr(Z_n \le z_\beta) = \beta$.

Comme la loi de Z_n

ne dépend pas de μ,

$$\forall \mu \in \mathbb{R} \quad P_{\mu}\left(z_{\frac{\alpha}{2}} \leq Z_n \leq z_{1-\frac{\alpha}{2}}\right)$$

$$= 1 - \alpha$$

Donc

$$1-\alpha = P_{\mu}\left(z_{\frac{\alpha}{2}} \leq \frac{\left(\overline{X}_n - \mu\right)\sqrt{n}}{} \leq z_{1-\frac{\alpha}{2}}\right)$$

$$= P_{\mu}\left(-\overline{X}_n + \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}} \leq -\mu \leq -\overline{X}_n + \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}}\right)$$

Made with Goodnotes

$$= P_\mu\left(\bar{Y}_n - \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}} \leq \mu \leq \bar{Y}_n - \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\right)$$

Par symétrie $z_{\frac{\alpha}{2}} = -z_{1-\frac{\alpha}{2}}$.

173

$$\left[\bar{Y}_n \pm \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}}\,\sigma\right] \quad \alpha = 0{,}05$$

$$= \left[5{,}34 \pm \frac{z_{0{,}975}}{\sqrt{64}=8}\,0{,}12\right]$$

$$\approx \left[5{,}34 \pm \frac{1{,}96}{8}\,0{,}12\right]$$

$\approx [5{,}37,\ 5{,}37]$

---

175 $\alpha = 0{,}05$

$$t_{n-1,1-\frac{\alpha}{2}} = t_{8,0.975}$$

$$= 2{,}306$$

$$\left[\bar{Y}_n \pm \frac{2{,}306}{\sqrt{n}} S_n\right]$$

$$= \left[1040 \pm \frac{2{,}306}{3} 16\right]$$

$$= [1027{,}8,\ 1052{,}2]$$

180 $Pr_\theta(T_n \leq t) = Pr_\theta(M_n \leq t\theta)$

$\underset{\text{ex } 12.1}{=} \left(\frac{t\theta}{\theta}\right)^n = t^n \quad \forall t \in [0,1]$

ne dépend pas de θ. Donc c'est un pivot. L'IC correspondant est $[M_n, \alpha^{-1/n} M_n]$

---

182 $L(\lambda) = e^{-n\lambda} \prod_{i=1}^{n} \frac{\lambda^{Y_i}}{Y_i!}$

(algèbre) $\hat{\lambda}_{ML} = \overline{Y}_n$

$\ell''(\lambda) = \frac{-n\overline{Y}_n}{\lambda^2}$ $J_n(\hat{\lambda}_{ML}) = \frac{n}{\overline{Y}_n}$

Made with Goodnotes

$$IC\left[\bar{Y}_n \pm z_{1-\frac{0.9}{2}}\sqrt{\frac{\bar{Y}_n}{n}}\right]$$

approximatif

$z_{0.95} \approx 1.64$ (table)

---

(183) $E_\theta[Y_i] = \frac{\theta}{2}$

$\mathrm{Var}_\theta[Y_i] = \frac{\theta^2}{12}$ (algèbre)

$$\underbrace{\sqrt{n}\,\frac{\bar{Y}_n - \frac{\theta}{2}}{\sqrt{\theta^2/12}}}_{T_n} \overset{app}{\sim} N(0,1)$$

(TCL)

Donc

$$1-\alpha \approx Pr_\theta\left(-z_{1-\frac{\alpha}{2}} \leq T_n \leq z_{1-\frac{\alpha}{2}}\right)$$

$$= Pr_\theta\left(\frac{-z_{1-\frac{\alpha}{2}}}{\sqrt{12n}} + \frac{1}{2} \le \frac{\bar{Y}_n}{\theta} \le \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{12n}} + \frac{1}{2}\right)$$

(algèbre)

$$= Pr_\theta\left(\frac{\bar{Y}_n}{\left(\frac{z_{1-\frac{\alpha}{2}}}{\sqrt{12n}}\right) + \frac{1}{2}} \le \theta \le \frac{\bar{Y}_n}{\left(\frac{-z_{1-\frac{\alpha}{2}}}{\sqrt{12n}}\right) + \frac{1}{2}}\right)$$

On obtient un IC approximatif, dont la longueur est $O\left(\frac{1}{\sqrt{n}}\right)$ alors que pour $[M_n, M_n \alpha^{-\frac{1}{n}}]$

la longueur est $o(\frac{1}{n})$
bcp plus court.

---

186 $z_i = \text{diff}$

$= (-2, 6, 6, 5, -1, \ldots)$

$\bar{z}_n \simeq 3{,}5 > 0$

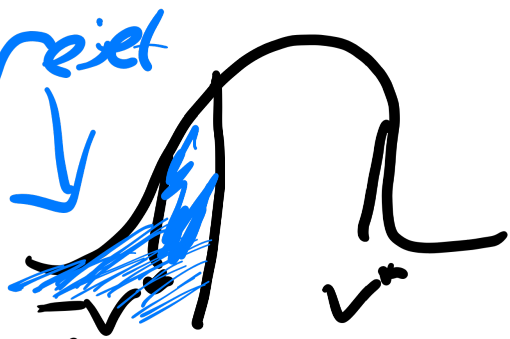
$z_i < 0$ 4 fois
$z_i > 0$ 10 fois

189

densité t_{n-1}

bilatéral

unilatéral



$-v^* < v_* < 0$

195

$$Pr_{H_0}(|T|>v^*)$$
$$= Pr_{H_0}(T<-v^*) + Pr_{H_0}(T>v^*)$$
$$= 2\,Pr_{H_0}(T>v^*)$$
$$= \alpha \iff Pr_{H_0}(T\le v^*) = 1-\frac{\alpha}{2}$$

$$t_{obs} = \frac{994 - 1000}{\sqrt{64{,}88/10}}$$

207

IC du pivot de la diapositive 180.

Pour $\alpha \in (0,1)$ on choisit $s>0$ tel que $\alpha = \mathbb{P}r(T_n \leq s)$. Donc

$$1-\alpha = \mathbb{P}r(T_n > s) = 1 - \mathbb{P}r(T_n \leq s)$$

$$= 1 - \mathbb{P}r_\theta\left(\frac{M_n}{\theta} \leq s\right) = 1 - \mathbb{P}r_\theta\left(\theta \geq \frac{M_n}{s}\right)$$

$$= \mathbb{P}r_\theta\left(\frac{M_n}{s} \geq \theta\right)$$

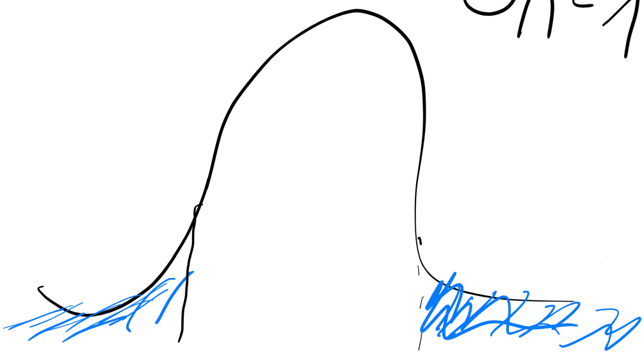
M_n continue

Pour trouver s on a $\alpha = \mathbb{P}r(T_n \leq s) = s^n$ donc $s = \alpha^{1/n}$ et on obtient IC $[-\infty, M_n \alpha^{-1/n}]$. or, forcément $\theta \geq M_n$ donc on peut prendre $[M_n, M_n \alpha^{-1/n}]$.

Remarque On aurait pu commencer par choisir t tel que $\alpha = \Pr(T_n \geq t)$. Cela amène à l'intervalle $[M_n(1-\alpha)^{-1/n}, \infty)$ qui est également un intervalle de confiance valide.

196

t_{n-1}



$-t_{obs}$ t_{obs} $Pr_{H_0}(|T| > |t_{obs}|)$

$$p_{obs} = Pr_{H_0}(|T| > |t_{obs}|)$$

$$= Pr_{H_0}(T > |t_{obs}|) + Pr_{H_0}(T < -|t_{obs}|)$$

symétrie

$$= 2 Pr(T < -|t_{obs}|)$$

$$= 2 F_{t_{n-1}}(-|t_{obs}|)$$

$$\overset{\text{symm}}{=} 2\left(1 - F_{t_{n-1}}(|t_{obs}|)\right)$$

202

$H_0: \mu = \mu_0 \quad H_1: \mu \neq \mu_0$

$$T_n^{\mu_0} = \sqrt{n}\,\frac{\bar{Y}_n - \mu_0}{S_n}$$

rejetter $H_0 \Longleftrightarrow |T_n^{\mu_0}| > t_{n-1, 1-\frac{\alpha}{2}}$

$\Longleftrightarrow \mu_0 \notin IC$ de 202

206

$$e_1 = e_2 = \dots = e_6 = 10$$

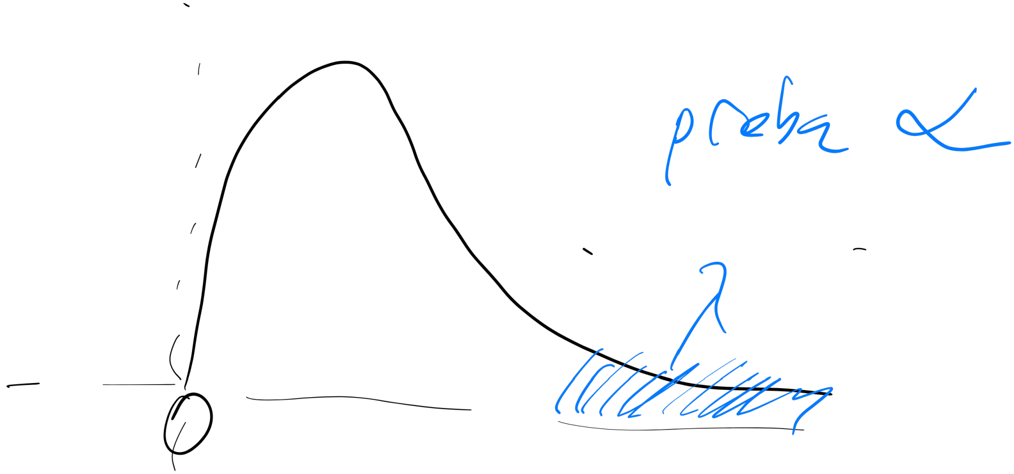
$$t_{obs} = \frac{(8-10)^2}{10} + \frac{(10-10)^2}{10} + \dots + \frac{(4-10)^2}{10} = 8.5$$

$$\nu = 6-1 = 5 \qquad \alpha = 0.05$$

$$1-\alpha = 0.95$$

$$\chi^2_{5,0.95} \simeq 11.1 > t_{obs}$$

pas de rejet à 5%



2a) $e_i =$

22,75, 135,91, 341,34

341,34, 135,91, 22,75

$$t_{obs} = \frac{(34-22{,}75)^2}{22{,}75} + \dots$$

$\boxed{\text{reiet}}$ — $(13{,}27)\ 11{,}1 = \chi^2_{6-1,\,0{,}95}$

210

$$t_{obs} = \frac{\left(32 - \frac{44 \times 52}{95}\right)^2}{\left(\frac{44 \times 52}{95}\right)} + \cdots + \frac{\left(9 - \frac{15 \times 43}{95}\right)^2}{\left(\frac{15 \times 43}{95}\right)}$$

$$\sim 10{,}67 > 5{,}99 = \chi^2_{2,\,0.95}$$

$$\nu = (2-1)(3-1) = 2 \qquad \boxed{\text{reject}}$$

220

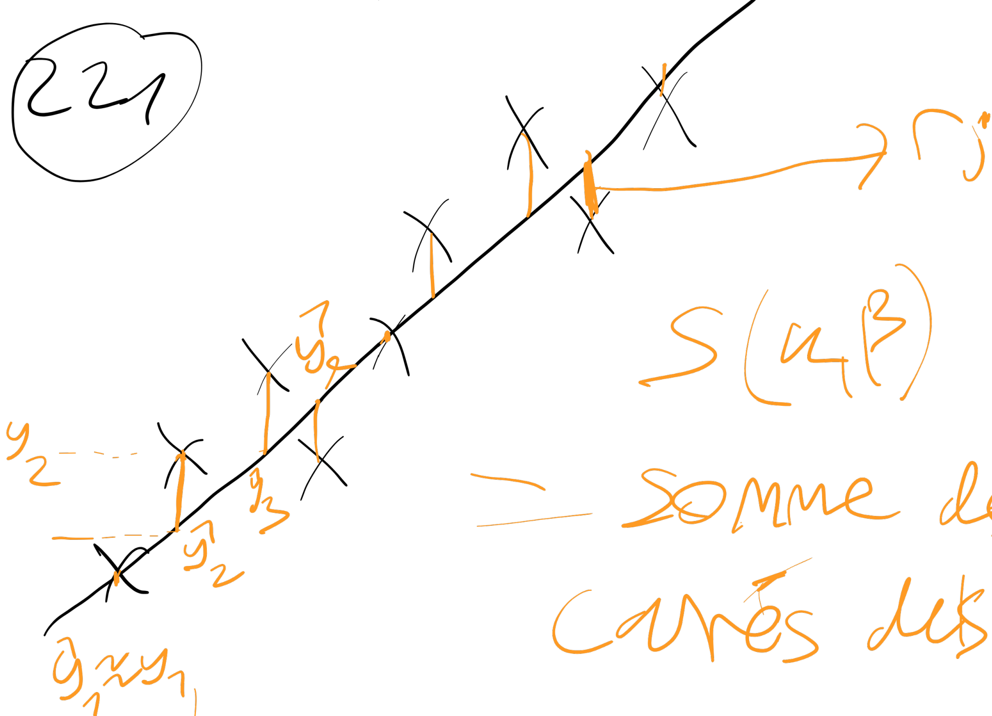
parabole $\mu(x) = a_0 + a_1 x + a_2 x^2$

→ estimer $a_0, a_1, a_2 \in \mathbb{R}$

linéaire $\mu(x) = \alpha + \beta x$

→ estimer $\alpha, \beta \in \mathbb{R}$

221



$S(\alpha, \beta)$ = somme des carrés des longueurs des lignes orange

$$S(\alpha,\beta) = \sum_{i=1}^{n} r_i^2$$

(223) $\frac{\partial S}{\partial a}(a,\beta) = 2na + 2n\beta\overline{x}_n - 2n\overline{y}_n$

$\hat{a}_n, \hat{\beta}_n$ sont telles que

$\frac{\partial S}{\partial a}(\hat{a}_n, \hat{\beta}_n) = 0 = \frac{\partial S}{\partial \beta}(\hat{a}_n, \hat{\beta}_n)$

$$\begin{cases} \hat{a}_n = \overline{y}_n - \hat{\beta}_n \overline{x}_n \\ 0 = n\hat{a}_n \overline{x}_n + \hat{\beta}_n \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i y_i \end{cases}$$

$= n\,\overline{x}_n \overline{y}_n - n\hat{\beta}_n \overline{x}_n^2 + \hat{\beta}_n \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i y_i$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{y}_n \overline{x}_n}{\sum_{i=1}^{n} x_i^2 - n\overline{x}_n^{\,2}}$$

$= (222)$ (exercice)

<u>indication</u> $\sum_{i=1}^{n} (x_i - \overline{x}_n)\overline{y}_n = 0$

224

$r = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} := \mathbf{1}_n$

$r \perp \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $r \perp \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \vec{y} \in \mathbb{R}^n$

$$\|r + (\tilde{y} - \mathbb{1}_n \bar{y}_n)\|^2$$

$$= \|r\|^2 + \|\tilde{y} - \mathbb{1}_n \bar{y}_n\|^2$$

---

231 $y_j \sim N(a + \beta x_j, \sigma^2)$

$$L(a, \beta, \sigma^2) =$$

$$\prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{(y_j - (a + \beta x_j))^2}{-2\sigma^2}\right)$$

après avoir trouvé a, β

il faut minimiser

$$g(\sigma^2) = -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} r_i^2$$

$$\mathbb{E}_{\sigma^2}\left[S_n^2\right] = \sigma^2$$

$$\frac{\partial g}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n} r_i^2$$

Attention ! on dérive par rapport à σ^2 non pas σ

$$\frac{\partial g}{\partial \sigma^2} = 0 \iff \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} r_i^2$$