Exercice 8.1 (a) [*2 marks*] Ω is the space of all permutations of the numbers $1, \ldots, n$. (These are equi-probable.)

(b) [*3 marks*] For Barthelemy's door to open at the rth attempt, the $r-1$ first keys chosen must have been wrong. At the first attempt, there is a probability $(n-1)/n$ of choosing the wrong key. At the second attempt, the probability is $(n-2)/(n-1)$, at the third attempt it is $(n-3)/(n-2)$, etc., and $(n-r+1)/(n-r+2)$ at the $(r-1)$st attempt, for $r = 2, \ldots, n$. The right key has to be chosen at the rth attempt, which has probability $1/(n-r+1)$. The probability that the door will open on the rth attempt is therefore equal to

$$\Pr(A_r) = \frac{n-1}{n} \times \frac{n-2}{n-1} \times \cdots \times \frac{n-r+1}{n-r+2} \times \frac{1}{n-r+1} = \frac{1}{n}, \quad r = 1, \ldots, n.$$

Equivalently one could argue by symmetry that the probability that the correct key is in place r in a permutation of n objects is $1/n$.

(c) [*2 marks*] Let X be the number of attempts made by Barthelemy ; we seek

$$\mathrm{E}(X) = \sum_{r=1}^{n} r \Pr(A_r) = n^{-1} \sum_{r=1}^{n} r = n^{-1} \times n(n+1)/2 = (n+1)/2.$$

(d) [*3 marks*] At each attempt, the probability that Bastien chooses the right key is $1/n$. The number of tries necessary therefore follows a geometric distribution of parameter $1/n$. The probability that the door will open on the rth attempt is then

$$\Pr(A_r) = \frac{1}{n} \left(\frac{n-1}{n} \right)^{r-1}, \quad r = 1, 2, 3, \ldots.$$

Exercice 8.2 (a)[*3 marks*] Let R and S denote respectively the run times of the first and second jobs. Since they are uniformly distributed from zero to three hours, their cumulative distribution functions are

$$F_R(z) = F_S(z) = \begin{cases} 0, & z \le 0, \\ \frac{z}{3}, & 0 < z < 3, \\ 1, & z \ge 3. \end{cases}$$

The cumulative distribution function of the longest run time $X = \max(R, S)$ is

$$\begin{aligned} F_X(x) = \Pr(X \le x) &= \Pr\{\max(R,S) \le x\} = \Pr(R \le x, S \le x) \\ &= \Pr(R \le x)\Pr(S \le x) = F_R(x) F_S(x), \end{aligned}$$

so,

$$F_X(x) = \begin{cases} 0, & z \le 0, \\ \frac{z^2}{9}, & 0 < z < 3, \\ 1, & z \ge 3, \end{cases}$$

and the probability density function of X is

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} 2x/9, & 0 < x < 3, \\ 0, & \text{elsewhere.} \end{cases}$$

(b)[*2 marks*] The expected waiting time until both processors are free is

$$\mathrm{E}(X) = \int_0^3 x\tfrac{2x}{9}\mathrm{d}x = \tfrac{2}{9}\tfrac{x^3}{3}\Big|_0^3 = 2 \text{ hours.}$$

(c)[*3 marks*] We want

$$\mathrm{E}(X \mid X > 1) = \int_1^3 x \times \frac{2x/9}{1 - F_X(1)}\,\mathrm{d}x = \frac{1}{1 - 1/9} \times \frac{2}{9}\left[\frac{x^3}{3}\right]_1^3 = 13/6.$$

(d)[*2 marks*] We want

$$\Pr(X < 1.5 \mid X < 2) = F_X(3/2)/F_X(2) = 9/16.$$

We also accept (following on from (c))

$$\Pr(X < 1.5 \mid 1 < X < 2) = \frac{F_X(3/2) - F_X(1)}{F_X(2) - F_X(1)} = 5/12.$$

Exercice 8.3 (a) [*2 marks*] Let $X \sim \mathcal{N}(10, 4^2)$ denote the average winter temperature, and note that $Z = (X - 10)/4 \sim \mathcal{N}(0, 1)$. Then

$$p = \Pr(X \le 4) = \Pr\{(X - 10)/4 \le (4 - 10)/4\} = \Pr(Z \le -1.5) = 1 - 0.9332 = 0.0668,$$

using the normal probability tables, and noting that $\Phi(-1.5) = 1 - \Phi(1.5)$.

(b) [*3 marks*] Let Y denote the number of years out of ten without extreme cold. Then $Y \sim B(10, 1-p)$, and the required probability is that $Y \in \{10, 9, 8\}$, i.e.,

$$\Pr(Y \ge 8) = (1 - p)^{10} + \binom{10}{1}(1 - p)^9 p^1 + \binom{10}{2}(1 - p)^8 p^2 = \cdots = 0.975.$$

Exercice 8.4 (a) [*3 marks*] Since the function $x \mapsto 1/x$ is monotonic for $x > 0$, we have

$$\Pr(Y \le y) = \Pr(1/X \le y) = \Pr(X \ge 1/y) = \exp(-\lambda/y), \quad y > 0, \lambda > 0,$$

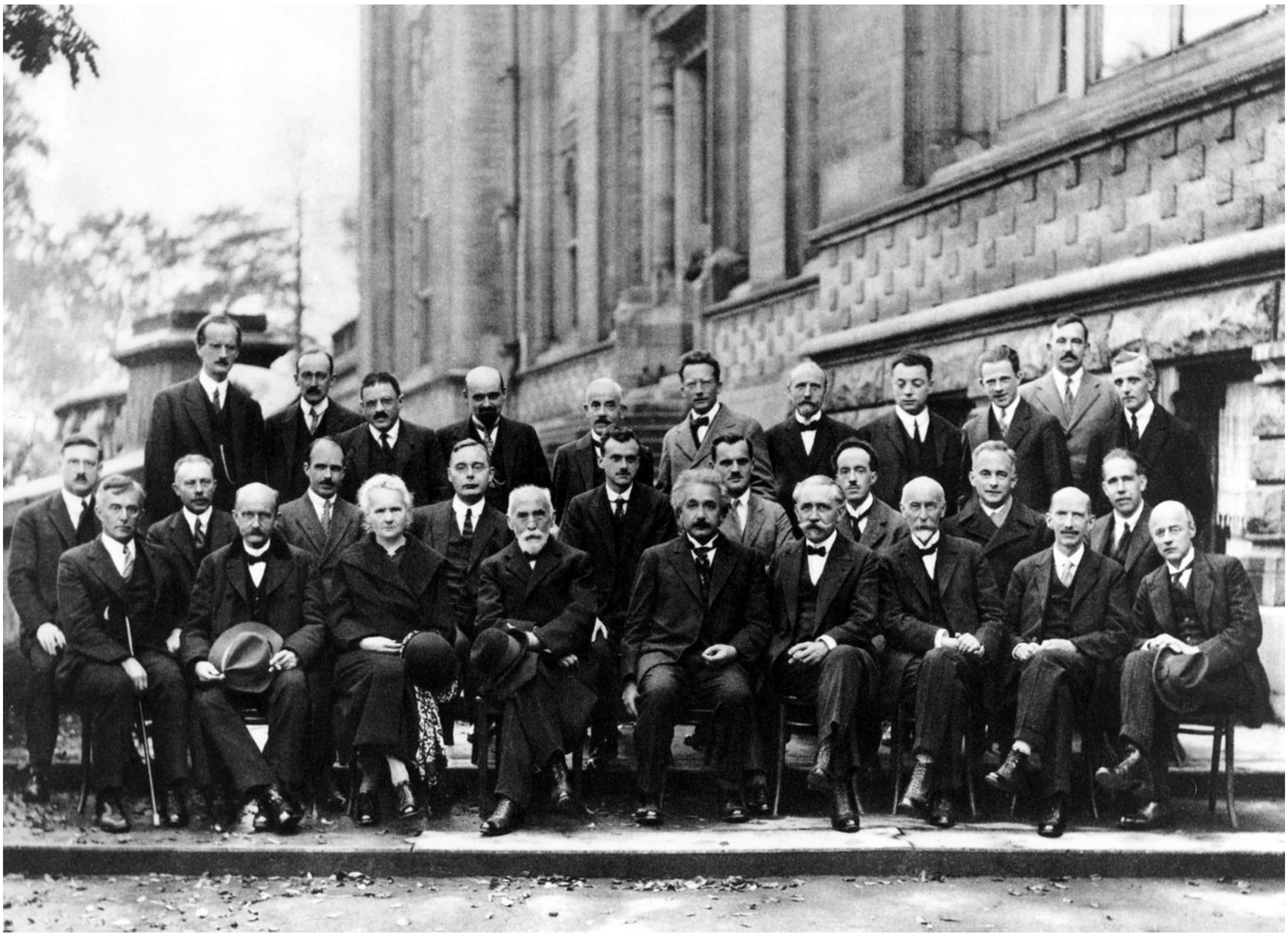
and clearly $\Pr(Y \le y) = 0$ for $y \le 0$. (Y has the Fréchet distribution.)

(b) [*2 marks*] Clearly $\mathrm{E}(Y) > 0$ if it exists, and since the exponential density exceeds $\lambda e^{-\lambda}$ for $t \in (0, 1)$, we have

$$\mathrm{E}(Y) = \mathrm{E}(1/X) = \int_0^\infty x^{-1}\lambda e^{-\lambda x}\,dx > \lambda \int_0^1 x^{-1}\,dx = \lambda e^{-\lambda}\left[\log x\right]_0^1 = +\infty.$$

Therefore $\mathrm{E}(Y)$ does not exist.

Exercice 8.5
[*5 or more marks*] This is a reference to discussions on the meaning of quantum mechanics at the third Solvay Conference in 1927 (and later, look up "Bohr–Einstein debates' in Wikipedia). In the photo (from Wikipedia), Max Planck is in the first row, second from the left, Albert Einstein is in the centre of the front row, Niels Bohr is at the right-hand end of the second row, and Werner Heisenberg is third from the right in the back row. Auguste Piccard is at the left-hand end of the back row. Of the 29 people present, 17 won Nobel Prizes ; the only woman present, Marie Curie, won them in both physics and chemistry.



In the (fortunately imaginary) duel, Albert reasons that he has three options for his first shot. He can aim at Werner, or at Niels, or he can fire into the ground. He knows that both opponents will aim to maximise their chances of survival, so if he shoots at one of them and misses, they will behave as if he had fired into the ground. Let S denote the event that Albert survives.

— If Albert fires into the ground, Werner will shoot at Niels. If Werner hits Niels, then he and Albert will have a shoot-out, with Albert firing first, but if Werner misses, he will certainly be killed by Niels, and then Albert has just one possible shot before he too is killed by Niels. Thus the probability that Albert survives is

$$\begin{aligned} P_{\text{ground}} &= \mathrm{Pr}(S \mid \text{W hits N})\mathrm{Pr}(\text{W hits N}) + \mathrm{Pr}(S \mid \text{W misses N})\mathrm{Pr}(\text{W misses N}) \\ &= P \times \frac{2}{3} + \frac{1}{3} \times 1 \times \frac{1}{3}, \end{aligned}$$

say, where (using the hint, see below)

$$P = 1/3/(1/3 + 2/3 - 2/9) = 3/7$$

is the probability that Albert wins a shoot-out with Werner, with Albert firing first. Hence $P_{\text{ground}} = 25/63$.

— If Albert shoots at Werner, then his survival probability is

$$\begin{aligned} P_{\text{Werner}} &= \Pr(S \mid \text{A hits W})\Pr(\text{A hits W}) + \Pr(S \mid \text{A misses W})\Pr(\text{A misses W}) \\ &= 0 \times \frac{1}{3} + P_{\text{ground}} \times \frac{2}{3} = \frac{2P_{\text{ground}}}{3} < P_{\text{ground}}. \end{aligned}$$

— If Albert shoots at Niels, then his survival probability is

$$\begin{aligned} P_{\text{Niels}} &= \Pr(S \mid \text{A hits N})\Pr(\text{A hits N}) + \Pr(S \mid \text{A misses N})\Pr(\text{A misses N}) \\ &= P \times \Pr(\text{W misses A}) \times \frac{1}{3} + P_{\text{ground}} \times \frac{2}{3} \\ &= \frac{3}{63} + \frac{25}{63} \times \frac{2}{3} \\ &= P_{\text{ground}} \left(\frac{2}{3} + \frac{3}{25} \right) < P_{\text{ground}}. \end{aligned}$$

Hence Albert's best option is to fire into the ground on his first shot.

— For the last part, note that Niels will survive if Werner misses him. If Niels then shoots Werner, and if Albert misses Niels, then Niels can shoot Albert and will be the winner. The probability of this sequence of events is

$$\Pr(\text{ Niels wins }) = \tfrac{1}{3} \times 1 \times \tfrac{2}{3} \times 1 = 2/9 = 14/63.$$

This implies that

$$\Pr(\text{ Werner wins }) = 1 - \Pr(\text{ Albert wins }) - \Pr(\text{ Niels wins }) = 1 - \tfrac{25}{63} - \tfrac{14}{63} = \tfrac{24}{63},$$

which is (just) smaller than the corresponding probability for Albert.

Alternatively, Werner will survive if he hits Niels, and then survives a shoot-out with Albert in which Albert fires first but ultimately loses. This outcome has probability $\frac{2}{3} \times (1 - 3/7) = 24/63$.

— For the hint, note that the first duellist will win on his nth shot if both he and his opponent have missed $(n-1)$ times and then he hits his opponent, and this happens with probability

$$\sum_{n=1}^{\infty} p\{(1-p)(1-q)\}^{n-1} = \frac{p}{1-(1-p)(1-q)} = \frac{p}{p+q-pq}.$$