PSI

Cours 5

$P(X < x) = ?? = P(X \leq x)$

Ex. 5.1 1. Soit Y = # diodes qui doivent être remplacées lors des premières 150 heures

2. $Y \sim Bin(n=5,\ p = P(X<150))$; Vérification des 4 conditions :
   (i) nombre fixe $(n=5)$
   (ii) épreuves de Bernoulli (2 possibilités : remplacée / pas)
   (iii) épreuves indéps (remplacements indéps. selon le problème)
   (iv) même probabilité de remplacement $= p(X<150)$

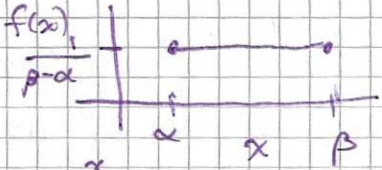
$$p(X<150) = \int_0^{150} f(x)\,dx = \int_0^{100} 0\,dx + \int_{100}^{150} \frac{100}{x^2}\,dx$$

$$= \frac{-100}{x}\Big|_{x=100}^{150} = \frac{-100}{150} - \left[-\frac{100}{100}\right] = 1 - \frac{100}{150} = 1 - \frac{2}{3} = \underline{\underline{\frac{1}{3}}}$$

$\Rightarrow Y \sim Bin(n=5,\ p=\frac{1}{3})$

3, 4. $P(Y=2) = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = 10 \times \frac{8}{243} = \underline{\underline{\frac{80}{243}}}$ (≈ 0.33) [loi binom., simp.]

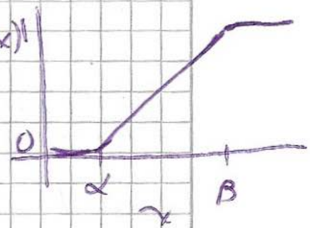
Ex. 5.2 $f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha < x < \beta \\ 0 & \text{sinon} \end{cases}$



$F(x) = \begin{cases} 0 & x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \alpha < x < \beta \\ 1 & x \geq \beta \end{cases}$

$\alpha < x < \beta \Rightarrow \int_\alpha^x f(x)\,dx = \int_\alpha^x \frac{1}{\beta-\alpha}\,dx$

$= \frac{x}{\beta-\alpha}\Big|_\alpha^x = \frac{x}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} = \underline{\underline{\frac{x-\alpha}{\beta-\alpha}}}$



Espérance pour VA uniforme

$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$ — déf. espérance

$= \int_\alpha^\beta \frac{x}{\beta-\alpha}\,dx$ — loi uniforme

$= \frac{\beta^2-\alpha^2}{2(\beta-\alpha)}$ — évaluation de l'intégrale

$= \frac{(\beta+\alpha)(\beta-\alpha)}{2(\beta-\alpha)} = \frac{\beta+\alpha}{2}$ — simplification

Variance de VA uniforme

$E[X^2] = \int_\alpha^\beta \frac{x^2}{\beta-\alpha}\,dx$ — thm. $E[g(X)]$, loi uniforme

$= \frac{\beta^3-\alpha^3}{3(\beta-\alpha)}$ — évaluation de l'intégrale

$= \frac{(\beta-\alpha)(\beta^2+\alpha\beta+\alpha^2)}{3(\beta-\alpha)} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$ — simplification

Ex. 5.3 $X \sim \mathcal{U}(0,1)$

a) $P(X<0.3) = \int_{x=0}^{0.3} 1 \cdot dx = x \Big|_0^{0.3} = 0.3 - 0 = \underline{\underline{0.3}}$

b) $P(X>0.6) = \int_{x=0.6}^{1} 1 \cdot dx = x \Big|_{0.6}^{1} = 1 - 0.6 = \underline{\underline{0.4}}$

c) $P(0.3<X<0.8) = \int_{x=0.3}^{0.8} 1 \cdot dx = x \Big|_{0.3}^{0.8} = 0.8 - 0.3 = \underline{\underline{0.5}}$

X **Ex. 5.6** $X \sim \mathcal{E}xp(\lambda) \Rightarrow f(x) = \lambda e^{-\lambda x}$

$\Rightarrow F(a) = P(X \leq a) = \int_0^a \lambda e^{-\lambda x} = -e^{-\lambda x}\Big|_0^a = -e^{-\lambda a} - [-\underbrace{e^{-0}}_{1}] = \underline{\underline{1 - e^{-\lambda a}, \ a \geq 0}}$

X **Ex. 5.7** 1. Soit X = # minutes pour servir le client

2. $X \sim \mathcal{E}xp(\lambda = 1/4)$ (car la moyenne $= 4 = 1/\lambda$)

a) 3, 4. $P(4<X<5) = \int_{x=4}^{5} \frac{1}{4} e^{-x/4} dx = -e^{-x/4}\Big|_4^5 = -e^{-5/4} + e^{-4/4}$

$= -0.2865 + 0.3679 \approx \underline{\underline{0.08}}$

b) $\Rightarrow$ Trouver la médiane de la distribution:

$\int_{x=0}^{c} \frac{1}{4} e^{-x/4} dx = 0.5 \Rightarrow 1 - e^{-c/4} = 0.5 \Rightarrow 0.5 = e^{-c/4}$

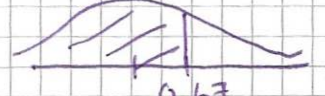
$\Rightarrow \log 0.5 = -c/4 \Rightarrow c = -4 \log 0.5 \approx$ <u>2.77 minutes</u>

5.4

Ex. 5.8 a) $P(2<X<5) = P\left(\frac{2-3}{\sqrt{9}} < \boxed{\frac{X-3}{\sqrt{9}}} < \frac{5-3}{\sqrt{9}}\right)$ [centrer-réduire]

$= P\left(-\frac{1}{3} < Z < \frac{2}{3}\right) = P(-0.33 < Z < 0.6)$ [simp., normale st.]

$= \Phi(0.67) - \Phi(-0.33)$



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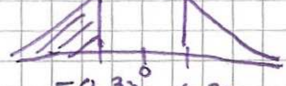
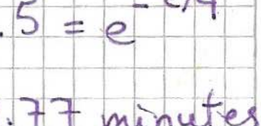


$\Rightarrow 0.7486 - 0.3707 = \underline{\underline{0.3779}}$ (≈ 0.38)

$\underbrace{0.67}_{0.7486 \text{ (table)}}$

$\underbrace{-0.33}_{*} \ 0 \ \underbrace{0.33}_{*}$ $* = * = 1 - \underbrace{\underbrace{\Phi(0.33)}_{0.6293}}_{0.3707}$

b) $P(X>0) = P\left(\boxed{\frac{X-3}{\sqrt{9}}} > \frac{0-3}{\sqrt{9}}\right) = P(Z>-1)$

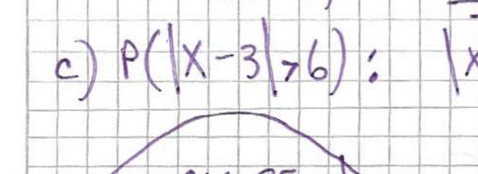


$= P(Z<1) = \underline{0.8413}$ (≈ 0.84)

c) $P(|X-3|>6)$: $|X-3|>6 \Rightarrow X-3>6$ ou $-(X-3)>6$

$X>9$ $\qquad -X>3 \Rightarrow X<-3$

$\Rightarrow \frac{X-3}{\sqrt{9}} = Z > \frac{9-3}{\sqrt{9}} = 2 \qquad \frac{X-3}{\sqrt{9}} = Z < \frac{-3-3}{\sqrt{3}} = -2$



≈ 0.95

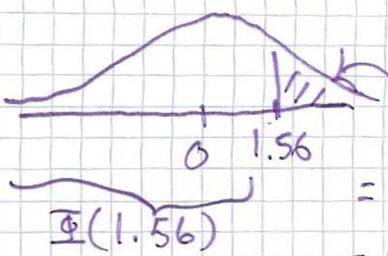
$(1-*)$ -2 0 2 $(1-*)$

$\Rightarrow (1-0.9772) + (1-0.9772) = 2\times(1-0.9772)$

$= \underline{0.9544}$ (≈ 0.95)

Ex. 5.5 a) $P(X>80) = P\left(\boxed{\frac{X-66}{9}} > \frac{80-66}{9}\right)$ [standardize]

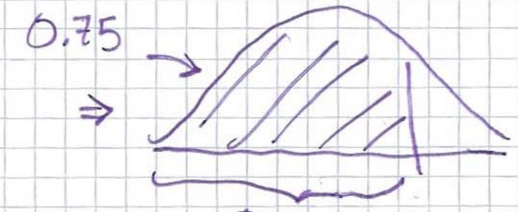
$= P(Z > 1.56)$ [simplify, standard normal]



$= 1-\Phi(1.56)$

$= 1-0.9406 \approx \boxed{0.06}$

b) $P(X<c) = 0.75 \Rightarrow P\left(\boxed{\frac{X-66}{9}} < \frac{c-66}{9}\right) = 0.75$



$\Rightarrow P\left(Z < \frac{c-66}{9}\right) = 0.75$

$\Phi(0.75) \approx \Phi(0.7486) = 1.67$

$\Rightarrow \frac{c-66}{9} = 0.67 \Rightarrow c = 66 + 0.67 \times 9 \approx \underline{\underline{72}}$