Exam 26 January 2022

Last name : ....

First name : ....

Sciper number : ....

- The duration of the exam is **three hours.**
- Please write **your name and your sciper number** on this sheet.
- Brain and pen only - no supporting materials or devices are allowed, no discussion is allowed.
- Black or blue pen only, no coloured writing, no pencils please!
- Please write your solutions on the **white paper** only. If you need extra space, there are numbered supplementary pages at the end of the booklet. If you need still more paper, please ask the proctors.
- Only white papers that are stapled on your booklet will be corrected. The loose coloured papers are for your rough work and will not be corrected.
- All four questions are **independent** and **carry roughly the same weight in total**. Also, the three parts of each question are marked independently.
- You may cite any basic result from the course (notes + exercises) without proof, unless asked to prove it.
- Please do not remove the stapling.

Q. 1	**Q. 2**	**Q. 3**	**Q. 4**

Total	**Mark**

Good luck!

Question 1

i Give the definition of a probability space, defining properly all its components. Give the definition of mutual independence of n collections of events, and of mutual independence of n random variables.

ii Define a probability space which contains four independent simple random walks S_1, S_2, S_3, S_4 of length n starting from 0. Show that each step of each walk is a random variable, and verify that the collection of all steps of all walks is a collection of mutually independent random variables. What is the probability that the walk S_1 ends at either n or $-n$ at step n, i.e. calculate $\mathbb{P}(|S_1(n)| = n)$.

iii Show that $\mathbb{P}(S_1(n) = S_2(n) = S_3(n) = S_4(n)) = O(n^{-3/2})$. Deduce that if we consider 4 independent random walks of infinite length then almost surely the three walks are equal only for finitely many times.

Question 2

i Give the definition of a discrete integrable random variable and its expectation. Calculate the expectation of a Poisson random variable.

ii Let X and Y be two discrete integrable random variables defined on the same probability space. Prove that $X + Y$ is also integrable.

iii Give an example of discrete random variables X, Y defined on the same probability space that are not integrable, but such that $X + Y$ is integrable. Does such an example exist when X, Y are independent? Refute with proof or give an example.

Question 3

i Give the definition of a random vector and of the covariance matrix of a random vector. Write down the density of a n dimensional Gaussian vector of mean μ and covariance matrix Σ.

ii Show that if (X_1, X_2) is a Gaussian vector then X_1, X_2 are independent if and only if $Cov(X_1, X_2) = 0$.

iii Let X_1, X_2 be independent Gaussians. Suppose that $aX_1 + bX_2$ is independent of $cX_1 + dX_2$ for some non-zero real numbers a, b, c, d. Find the ratio of variances of X_1 and X_2.

Question 4

i Let $X_0, X_1, X_2, \dots$ be a sequence of random variables defined on a common probability space. Explain why the event $E = \{X_n \text{ does not converge}\}$ is a measurable event. Give a definition of what it means for the sequence $X_1, X_2, \dots$ to converge to X_0 almost surely and what it means for a sequence of random variables to converge in probability.

ii State and prove the Strong Law of Large Numbers for i.i.d. random variables with finite forth moments.

iii Is there an $m > 0$ such that for any i.i.d. random variables $X_1, X_2, \dots$, we have that $\frac{1}{n^m}(X_1 + \cdots + X_n)$ converges to 0 in probability? Prove the claim or give a counterexample for each m.

(Supplementary page 1)

(Supplementary page 3)