Exam 3 February 2022

Last name : .....

First name : .....

Sciper number : .....

- The duration of the exam is **three hours.**
- Please write **your name and your sciper number** on this sheet.
- Brain and pen only - no supporting materials or devices are allowed, no discussion is allowed.
- Black or blue pen only, no coloured writing, no pencils please!
- Please write your solutions on the **white paper** only. If you need extra space, there are numbered supplementary pages at the end of the booklet. If you need still more paper, please ask the proctors.
- Only white papers that are stapled on your booklet will be corrected. The loose coloured papers are for your rough work and will not be corrected.
- All four questions are **independent** and **carry roughly the same weight in total**. Also, the three parts of each question are marked independently.
- You may cite any basic result from the course (notes + exercises) without proof, unless asked to prove it.
- Please do not remove the stapling.

Q. 1	**Q. 2**	**Q. 3**	**Q. 4**

Total	**Mark**

Good luck!

Question 1

i Give the definition of a probability space and explain what we mean by an event. Also give a definition of pairwise independence and mutual independence for 3 events E, F, G. Find an example of events E, F, G that are pairwise independent but not mutually independent.

ii Let E, F, G be pairwise independent events of positive probability defined on the same probability space. Suppose that F, G are conditionally independent given E. Prove that E, F, G are mutually independent.

iii Let X, Y be two independent discrete random variables defined on the same probability space. Suppose further that X and $-X$ have the same law, X is not almost surely a constant and XY and Y are independent. Find all possible probability laws for Y.

Question 2

i Give the definition of the expectation $\mathbb{E}X$ of a discrete random variable X. Calculate the expectation of a $Bin(n, p)$ random variable. Write down the expression for the expectation of a continuous random variable X with density f_X and also for the expectation of $g(X)$ when g is continuous (you may assume $g(X)$ is an integrable random variable).

ii Write down the density of the standard Gaussian random variable N. Let g be a smooth function $g : \mathbb{R} \to \mathbb{R}$. Explain why $g'(N)$ and $Ng(N)$ are random variables. Assume now that both of them are integrable. Prove that $\mathbb{E}g'(N) = \mathbb{E}(Ng(N))$.

iii Now suppose that Y is a random variable such that its moment generating function $M_Y(t)$ exists for all $t \in \mathbb{R}$. Suppose further that for all smooth functions $g : \mathbb{R} \to \mathbb{R}$ such that $g'(Y)$ and $Yg(Y)$ are integrable, it holds that $\mathbb{E}g'(Y) = \mathbb{E}(Yg(Y))$. Prove that Y has the law of a standard Gaussian.

Question 3

i Let $X_0, X_1, X_2, \ldots$ be a sequence of random variables defined on a common probability space. Explain why the event $E = \{X_n \text{ converges to } X_0\}$ is a measurable event. Give a definition of what it means for the sequence $X_1, X_2, \ldots$ to converge to X_0 almost surely and what it means for a sequence of random variables to converge in law.

ii State and prove the first Borel-Cantelli lemma.

iii Prove that for any sequence $X_1, X_2, \ldots$ of random variables, one can find a sequence of real numbers $c_1, c_2, \ldots$ such that X_n/c_n converges to 0 almost surely.

iv Now suppose that $X_1, X_2, \ldots$ are positive almost surely and that X_n converges in law. Can you always find a sequence of real numbers $(c_n)_{n\geq 1}$ such that as $n \to \infty$, the sequence X_n/c_n converges in law to an almost surely positive random variable ?

Question 4

Zitzerland has five big towns: Ausanne, Bausanne, Causanne, Daussane and Vausanne and a very powerful railway company ZBB. ZBB has built a two-sided railway track between each pair of big towns. The new director is environmentally conscious and realises that the traffic has to be reduced. She hesitates between two regimes: 1) either every day each track is turned on independently with probability 1/2 and off with the same probability or 2) each day all tracks are turned on simultaneously with probability p or off with probability $1 - p$. Find a suitable probability space to analyse and answer the following questions.

i What probability p should the director choose so that in average the same number of tracks is turned on in both regimes.

ii For this choice of p, and for both regimes 1 and 2, what is the probability that the director can not travel anywhere from Vausanne next Monday?

iii For this choice of p, and for both regimes 1 and 2, what is the expected number of days that the director can travel from Vausanne to Ausanne during the next working week (5 days)? [She doesn't mind taking indirect routes and she always returns to home in Vausanne in the evening.]

iv For this choice of p, and for both regimes 1 and 2, what is the probability that the director can visit all towns during the next working week (5 days), assuming that she can take one trip per day and she starts in Vausanne on Monday? [She does not need to visit the towns in a fixed order and she sleeps in the town where she is.]

v Which regime would you personally recommend for the director to choose? Justify!

(Supplementary page 1)

(Supplementary page 3)