Worksheet #9

Metric and Topological Spaces

November 12, 2024

Problem 1. *Let (X,τ) be a topological space which is sequentially compact (Definition 3.4 in the notes). If $Y \subset X$ is closed, prove that Y is also sequentially compact (w.r.t. the subspace topology).*

Problem 2. *Prove that any compact and Hausdorff topological space (X,τ) is normal (Definition 3.17 in the notes).*

Problem 3. *Let (M,d) be a metric space. Prove that the following statements are true.*

a) $d : M \times M \to \mathbb{R}$ *is continuous, where on $M \times M$, we consider the product topology induced by the metric topology on M, and on $\mathbb{R}$, we consider the usual topology.*

b) The metric topology on M is the coarsest topology on M making d continuous.

c) Given $\emptyset \neq A \subset M$, the function $d(-,A) : x \mapsto \inf_{y \in A}\{d(x,y)\}$ is continuous.

d) The function in item c) is Lipschitz. Hence, it is uniformly continuous (for the definitions, see the bonus problems from worksheet 8).

e) If $\emptyset \neq A \subset M$ is compact, then for any $x \in M$ we can find $y \in A$ such that $d(x,A) = d(x,y)$. Here, you should use item c).

Problem 4. *Determine which of the following subsets of $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ are compact (usual topology).*

(i) $\mathrm{O}(n,\mathbb{R}) = \{P \in \mathrm{M}_n(\mathbb{R}) \,:\, P^tP = I_n\}$.

(ii) The set of matrices $A \in \mathrm{M}_n(\mathbb{R})$ such that $A^2 = I_n$.

(iii) The set of diagonalizable matrices in $\mathrm{M}_n(\mathbb{R})$.

(iv) The set of matrices $A \in \mathrm{M}_n(\mathbb{R})$ such that $A^2 - 3A + 2I_n = 0$.

Problem 5. *Let (X,τ) be a locally compact and Hausdorff topological space. Prove that the one-point compactification of (X,τ) is also Hausdorff.*

Problem 6. *Prove that for any $n \geq 1$, the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the sphere $S^n = \{x \in \mathbb{R}^n \,;\, |x| = 1\}$ (usual topologies).*