Worksheet #7

Topology I - point set topology

October 29, 2024

In this worksheet, if the topology in $\mathbb{R}^n$ is not specified, we are considering the usual Euclidean topology. Problems marked with a $(*)$ are optional; for those, you will need the following definitions.

Definition 1. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces. We say that two continuous functions $f, g : X \to Y$ are homotopic if there exists a family of continuous functions $h_t : X \to Y$ index by $t \in [0, 1]$ such that $h_0 = f$, $h_1 = g$ and $H : X \times [0, 1] \to Y$ given by $(x, t) \mapsto h_t(x)$ is continuous. We then say that the family $\{h_t\}$ is a homotopy from f to g.*

Definition 2. *A deformation retraction of a topological space (X, τ) onto a subspace A (with the subspace topology) is a homotopy from the identity on X to a retraction $r : X \to A$ (see Problem 4 below).*

Problem 1. *Are the following topological spaces simply-connected? Justify your answer.*

(a) $\mathbb{R}^n$ *with the usual Euclidean topology,*

(b) $\mathbb{R}^n \backslash \{0\} \subset \mathbb{R}^n$,

(c) any set X with the indiscrete topology,

(d) any convex subset of $\mathbb{R}^n$,

(e) $\mathbb{S}^1 \backslash \{(0, 1)\} \subset \mathbb{R}^2$

(f) the space Y from problem 7 in worksheet # 4.

Problem 2. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \to Y$ be a continuous map. Pick $x \in X$ and $y \in Y$ with $f(x) = y$. Prove that $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ defined by $f_*([\gamma]) = [f \circ \gamma]$ is a homomorphism of groups. Now, suppose that f is a homeomorphism. Prove that f_* is an isomorphism.*

Problem 3. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and pick $x \in X, y \in Y$. Prove that we have an isomorphism of groups $\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$.*

Problem 4 (Retraction)**.** *Let (X, τ_X) be a topological space and let $A \subseteq X$. Consider A with its subspace topology and assume that there exists $r : X \to A$ a continuous map such that $r_{|A} = id_A$ (any such map r is called a retraction of X onto A). Pick $a \in A$, let $\iota : A \to X$ denote the inclusion and prove that if $\iota_*(\pi_1(A, a))$ is a normal subgroup of $\pi_1(X, a)$, then $\pi_1(X, a) \simeq ker(r_*) \times im(\iota_*)$ as groups.*

Problem 5. *Consider the unit circle $\mathbb{S}^1$ as a subspace of $\overline{B(0, 1)}$, where $\overline{B(0, 1)} \subseteq \mathbb{R}^2$ denotes the closed unit ball with the subspace topology, and on $\mathbb{R}^2$ we consider the usual Euclidean topology. Using the fact that $\pi_1(\mathbb{S}^1) = \mathbb{Z}$, prove that there cannot exist a continuous map $\overline{B(0, 1)} \to \mathbb{S}^1$ that keeps $\mathbb{S}^1$ fixed point-wise.*

Problem 6 $(*)$**.** *Show that the map $H : \mathbb{C}^\times \times [0, 1] \to \mathbb{S}^1$ given by $(\lambda, t) \mapsto h_t(\lambda) \doteq \dfrac{\lambda}{(1 - t) + t \cdot |\lambda|}$ defines a deformation retraction of $\mathbb{C}^\times$ onto $\mathbb{S}^1$. Deduce that $\pi_1(\mathbb{C}^\times) = \pi_1(\mathbb{S}^1) = \mathbb{Z}$. Here, we are considering the usual topologies.*

Problem 7 $(*)$**.** *Let $n > 2$ and $k \leq n - 2$. Let $V = \mathbb{R}^n$ and $W = \{(v_1, \ldots, v_n) \in V \,;\, v_j = 0 \,\forall\, j \geq k + 1\}$. Construct a deformation retract of $X = V \backslash W$ onto $A = \{(v_1, \ldots, v_n) \in V \backslash \{0\} \,;\, v_i = 0 \,\forall\, i \leq k\}$. Deduce that $\pi_1(V \backslash W) = \mathbb{Z}$ if $n - k = 2$ and otherwise, $\pi_1(V \backslash W) = \{0\}$.*