Worksheet #6

Topology I - point set topology

October 15, 2024

Problem 1. *Consider $\mathbb{R}^2$ with its standard Euclidean topology.*

(a) Let $S \subset \mathbb{R}^2$ be finite and prove that $A = \mathbb{R}^2 \backslash S$ is path-connected.

(b) Give an example of a subset $S \subset \mathbb{R}^2$ such that $\mathbb{R}^2 \backslash S$ is not connected.

(c) Give an example of a path-connected subset $S \subset \mathbb{R}^2$ such that $cl(S)$ is not path-connected.

Problem 2. *Consider $\mathbb{R}$ and $\mathbb{R}^2$ with their standard Euclidean topology. Prove that $\mathbb{R} \backslash \{*\}$ is disconnected, while $\mathbb{R}^2 \backslash \{*\}$ is connected. Deduce that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Here $\{*\}$ simply means a point.*

Problem 3. *Justify the following assertion: "Every open interval $I \subset \mathbb{R}$ is connected since it is homeomorphic to $\mathbb{R}$"*

Problem 4. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f([a, b]) \subset [a, b]$. Prove that the line $y = x$ intersects the graph of f at least once. Here, we are considering the usual Euclidean topology.*

Problem 5. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces such that (X, τ_X) is connected and let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. Consider $X \times Y$ with its product topology. Prove that $G = \{(x, f(x)) : x \in X\} \subseteq X \times Y$ is connected with the subspace topology.*

Problem 6. *Let (X, τ) be a topological space. Let $A \subset X$ be a connected subset and $B \subset X$ be a subset both open and closed. Assume that $A \cap B \neq \emptyset$. Prove that $A \subset B$.*

Problem 7. *Prove that the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 ; x^2 + y^2 = 1\}$ is a connected subspace of $\mathbb{R}^3$ (usual Euclidean topology on $\mathbb{R}^3$).*

Problem 8. *Let (X, τ) be a topological space. Define a relation on X by setting $x \sim y$ if and only if there exists a path (on X) from x to y. Prove that this defines an equivalence relation. Conclude that (X, τ) is path-connected if and only if the quotient $(X/\sim)$ consists of a single point.*