

Worksheet #5

Topology I - point set topology

October 8, 2024

Problem 1. Let $M_n(\mathbb{R})$ denote the set of square $n \times n$ matrices. Identify this set with $\mathbb{R}^{n^2}$ to endow it with the usual Euclidean topology. Let $GL(n, \mathbb{R})$ denote the set of invertible matrices in $M_n(\mathbb{R})$. Prove that $GL(n, \mathbb{R})$ is open and not connected.

Problem 2. One can show that a metric space (M, d) is connected (w.r.t. the metric topology) if and only if any two points $a, b \in M$ are contained in some connected subspace of M . Use this fact to prove every real normed vector space is connected (w.r.t. the induced metric topology).

Problem 3. Let (X, τ) be a topological space. Let $A \subset X$ be a connected subspace. Prove that $\text{cl}(A)$ is also connected.

Problem 4. Let V be a real vector space of dimension n endowed with the Euclidean norm, and consider it as a topological space with the induced metric topology. Let W be a sub-vector space of dimension $d = n - 1$. Prove that $V \setminus W$ is not connected.

Problem 5. Consider $\mathbb{R}$ with the usual Euclidean topology, pick any two distinct real numbers, say a and b . Prove that $\{a, b\} \subset \mathbb{R}$ is not connected.

Problem 6. Consider $\mathbb{R}$ with the usual Euclidean topology. Prove that the following statements are true.

(b) $\mathbb{Q} \subset \mathbb{R}$ is not connected.

(b) The only (non-empty) connected subsets of $\mathbb{Q}$ are the singletons.

Use part (b) to conclude that every continuous function $f : \mathbb{R} \rightarrow \mathbb{Q}$ is constant.

Problem 7. Let (X, τ) be a topological space and assume that all connected components of X are open. Let $q : X \rightarrow Y$ be a quotient map. Prove that the connected components of Y are also open.

Problem 8. Let X be a set and let $\tau, \tilde{\tau}$ be two topologies on X . Assume that $\tau \subset \tilde{\tau}$. Is it true or false that if $A \subset X$ is disconnected with respect to τ , then A is disconnected with respect to $\tilde{\tau}$? If true, provide proof, and if false, provide a counter-example. What about the converse?