Worksheet #3

Topology I - point set topology

September 24, 2024

Problem 1. *Consider* $\mathbb{N}$ *with its cofinite topology. Give an example of a sequence that converges simultaneously to all* $n \in \mathbb{N}$.

Problem 2. *Let* $f : (X, \tau_X) \to (Y, \tau_Y)$ *be a function between two topological spaces. Prove that* f *is continuous if and only if for any subset* $A \subseteq X$*, we have that* $f(cl(A)) \subset cl(f(A))$.

Problem 3. *Consider* $X = \{0, 1\}$ *with the topology* $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$*. Given any subset* $A \subset X$*, define the function* $f_A : X \to \{0, 1\}$ *to be equal to* 1 *if* $x \in A$ *and equal to* 0 *otherwise. Prove that* f_A *is continuous if and only if* A *is open.*

Problem 4. *Let* $\emptyset \neq X$ *be a set, and endow it with the discrete metric* $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. *Prove that the topology induced by such metric is the discrete topology.*

Problem 5. *Let* $X = \{1, 2, 3, 4\}$ *and consider the topology (on* X*) given by* $\{\emptyset, X, \{1\}, \{2, 3, 4\}\}$*. Consider the function* $f : X \to X$ *given by* $f(2) = f(3) = f(4) = 1$ *and* $f(1) = 4$*. Prove that* f *is continuous.*

Problem 6. *Let* τ_E *(resp.* τ_d*) be the Euclidean (resp. discrete) topology on* $\mathbb{R}$*. Prove that the function* $f : (\mathbb{R}, \tau_E) \to (\mathbb{R}, \tau_d)$ *given by* $x \mapsto x^2$ *is not continuous.*

Problem 7. *Consider* $\mathbb{R}$ *and* $\mathbb{R}^2$ *with the usual Euclidean topology and pick* $f : \mathbb{R}^2 \to \mathbb{R}$ *continuous. Let* $A = \{x \in R^2\,;\, f(x) < 0\}$ *and* $B = \{x \in R^2\,;\, f(x) \leq 0\}$*. Prove that* $cl(A) \subset B$ *and* $A \subset int(B)$.

Problem 8. *Let* $f : (X_1, \tau_1) \to (X_2, \tau_2)$ *be a homeomorphism and pick* $\emptyset \neq A \subset X_1$*. Prove that* $x \in int(A) \iff f(x) \in int(f(A))$.