Worksheet #2

Topology I - point set topology

September 17, 2024

Key concepts

Definition 1. *Let X be a set. A* ***topology*** *on X is a collection $\mathcal{T}$ of subsets of X which satisfies the following properties:*

(T1) The sets $\emptyset$ and X are elements of $\mathcal{T}$.

(T2) If $\{U_i\}_{i\in I}$ is an arbitrary collection of elements of $\mathcal{T}$, then $\displaystyle\bigcup_{i\in I} U_i$ is also an element of $\mathcal{T}$.

(T3) If U and V are elements of $\mathcal{T}$, then so is their intersection $U \cap V$.

A set X endowed with a topology $\mathcal{T}$ is called a ***topological space****, and the elements of $\mathcal{T}$ are called the* ***open subsets*** *of X.*

Definition 2. *Let M be a set. A* ***metric*** *on M is a function $d : M \times M \to \mathbb{R}_{\geq 0}$ which satisfies the following properties for all $p, q, r \in M$:*

(M1) $d(p, q) = 0 \iff p = q$

(M2) $d(p, q) = d(q, p)$

(M3) $d(p, q) \leq d(p, r) + d(r, q)$

A set M endowed with a metric d is called a ***metric space****.*

Definition 3. *Let $X \neq \emptyset$ be an arbitrary set. A collection $\mathcal{B}$ of subsets of X is called* ***a basis for a topology*** *on X if and only if*

(i) given $x \in X$, we can find $B \in \mathcal{B}$ such that $x \in B$,

(ii) if $x \in B_1 \cap B_2$, with $B_i \in \mathcal{B}$, then we can find $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Given X and $\mathcal{B}$ as above,

$$\tau^{\mathcal{B}} := \{U \subset X \,;\, x \in U \Rightarrow \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}$$

is a topology on X and we say $\mathcal{B}$ is a basis for $\tau^{\mathcal{B}}$.

Exercises

Problem 1. *Let X be a set.*

(i) *Let $\mathcal{T} = \{\emptyset, X\}$. Prove $\mathcal{T}$ is a topology on X.*

(ii) *Let $\mathcal{T} = \mathcal{P}(X)$ be the collection of all subsets of X. Prove $\mathcal{T}$ is a topology on X.*

Problem 2. *Let $X = \{a, b\}$. Find all possible topologies on X.*

Problem 3. *Verify the set $\mathcal{T} = \{A \subset \mathbb{R} \,|\, \mathbb{R} \backslash A \text{ is finite or } A = \emptyset\}$ is a topology on $\mathbb{R}$.*

Problem 4. *Let $(X, \mathcal{T})$ be a topological space. Prove a subset $V \subset X$ is open if and only given $x \in V$ we can find $U \in \mathcal{T}$ such that $x \in U$ and $U \subset V$.*

Problem 5. *Let $(X, \mathcal{T})$ be a topological space. We say $D \subset X$ is **dense** (in X) if and only if for every $\emptyset \neq U \in \mathcal{T}$ we have $U \cap D \neq \emptyset$. If $A, B \in \mathcal{T}$ are dense (in X) prove $A \cap B$ is also dense (in X).*

Problem 6. *Let (M, d) be a metric space. Prove the following functions are also metrics on M:*

(i) $f(p, q) = \min\{1, d(p, q)\}$

(ii) $g(p, q) = \sqrt{d(p, q)}$

(iii) $h(p, q) = \dfrac{d(p, q)}{1 + d(p, q)}$

Problem 7. *Let $M = \mathcal{C}([0, 1])$ be the space of all continuous functions $f : [0, 1] \to \mathbb{R}$. Prove the functions below are metrics on M:*

(i) $d_1(p, q) = \sup\limits_{x \in [0,1]} |p(x) - q(x)|$

(ii) $d_2(p, q) = \displaystyle\int_0^1 |p(x) - q(x)| \,\mathrm{dx}$

Problem 8. *Prove that the collection $\mathcal{B}$ of balls $B(x, \delta)$ where $x \in \mathbb{Q}^n$ and $\delta \in \mathbb{Q} \cap (0, \infty)$ is a basis for the Euclidean topology on $\mathbb{R}^n$ (i.e., the Euclidean topology agrees with $\tau^{\mathcal{B}}$) and that there are countably many elements on this basis.*

Problem 9 (Kuratowski's formulation of a topological space)**.** *In 1922, Kuratowski considered an arbitrary set X with an operation named "closure" $\overline{cl}$ from the set of subsets of X to itself. We assume it to satisfy the following axioms:*

(K1) *For any two subsets A, B of X, we have $\overline{cl}(A \cup B) = \overline{cl}(A) \cup \overline{cl}(B)$;*

(K2) $A \subseteq \overline{cl}(A)$;

(K3) $\overline{cl}(\emptyset) = \emptyset$ *and* $\overline{cl}(X) = X$.

(i) *Show that if we start from a topological space (X, τ), then the closure defined in class satisfies all these conditions.*

(ii) *Suppose we have a set X and the operation $\overline{cl}$ in the opposite direction. Define a set C to be closed if it equals its closure. Show that the set $\{X \backslash C\,;\, C = \overline{cl}(C)\}$ is a topology on X.*