Worksheet #14

Practice problems for the exam

Topology I - Fall 2024

Problem 1. *Indicate whether the statements that follow are true or false. If true, provide a proof. If false, provide an explanation or a counter-example. Below* (X, τ_X) *and* (Y, τ_Y) *are topological spaces.*

(1) If $A \subset X$ and $B \subset Y$ are closed subsets, then $A \times B$ is closed in $X \times Y$, where on $X \times Y$ we consider the product topology.

(2) Let $f : X \to Y$ be a continuous map. If X is path-connected and compact then so is $f(X)$.

(3) Let A and B be subsets of X. If A and B are connected, then so is $A \cap B$ and $\mathrm{cl}(A) \cup \mathrm{cl}(B)$.

(4) If X is Hausdorff, then $X \times X$ is Hausdorff when given the product topology.

(5) For any $A \subset X$ we have that $\mathrm{cl}(X \backslash A) = X \backslash \mathrm{cl}(A)$.

(6) The interior of any finite subset $A \subset X$ is always empty.

(7) For any $x \in X$ the singleton $\{x\}$ is closed.

(8) If τ_X is the discrete topology, then X is disconnected.

(9) If τ_X is the indiscrete topology, then any sequence in X converges.

(10) If $c > b$, $X = [a, b] \cup [c, d]$ and τ_X is the usual topology, then $[a, b] \in \tau_X$.

(11) If $A \subset X$, then $\mathrm{int}(\mathrm{cl}(A)) = A$.

(12) If $f : X \to Y$ is continuous, and $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.

(13) If $f : X \to Y$ is continuous at a point $x \in X$ and $V \in \tau_Y$ contains $f(x)$, then $f^{-1}(V) \in \tau_X$.

(14) If τ_X is not the discrete topology, then there cannot exist $A \subset X$ such that the subspace topology on A is the the discrete topology (on A).

(15) If (X, τ_X) is disconnected, then any subspace of X is also disconnected.

(16) If (X, τ_X) is compact, then any closed subspace of X is also compact.

(17) If A and B are connected subspaces of X and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

(18) If A and B are connected subspaces of X and $A \cap B \neq \emptyset$, then $A \cap B$ is connected.

(19) If $X = \mathbb{Z}, Y = \mathbb{Z}^2$ and τ_X and τ_Y are the corresponding discrete topologies, then (X, τ_X) and (Y, τ_Y) are homeomorphic.

(20) If $f : X \to Y$ is continuous and X is compact and connected, then $f(X)$ can be non-compact and it can also be disconnected.

(21) If $X = \mathbb{R}$ and τ_X is the cofinite topology, then the function $f : X \to X$ given by $f(x) = \sin(x)$ is continuous.

(22) If $A \subset X$ is closed, then A will also be closed with respect to any topology that is coarser (smaller) than τ_X.

(23) If X is Hausdorff, then limits of sequences in X are unique.

(24) If τ_X and τ_Y are the discrete topologies on X and Y, respectively, then (X, τ_X) and (Y, τ_Y) are homeomorphic if and only if $|X| = |Y|$.

(25) If $A \subset X$ is such that $\mathrm{cl}(A)$ is connected, then $\mathrm{int}(A)$ is connected.

(26) If X is connected, then X will also be connected with respect to any topology that is coarser than τ_X.

(27) If (X, τ_X) is path-connected, then we can always find a continuous surjection

$$f : (X, \tau_X) \to (\{0, 1\}, \tau_{\mathrm{disc}}).$$

(28) If τ_X is the cofinite topology, then X is compact.

(29) If τ_X is the indiscrete topology and $\emptyset \neq A \subset X$, then A is dense.

(30) If τ_X is the discrete topology, then (X, τ_X) is not compact.

(31) If there exists a metric d on X such that the induced metric topology is τ_X, then all Cauchy sequences converge in (X, τ_X).

(32) If $X = \mathbb{R}^n$ and τ_X is the usual topology, then (X, τ_X) is a Baire space.

(33) If there exists a metric d on X such that the induced metric topology is τ_X and X is compact, then (X, d) is a Baire space.

(34) If $f : X \to Y$ is continuous with τ_X the discrete topology, then the subspace topology on $f(X) \subset Y$ is the discrete topology.

(35) If there exists a metric d on X such that the induced metric topology is τ_X, then every compact subset of X is closed and bounded.

Problem 2. *In this problem, you must justify your answers, and your examples should be explicit. Please provide an example of each of the following.*

(1) A topological space (X,τ) which is not Hausdorff and such that it has a point p satisfying that $X\backslash\{p\}$ is Hausdorff.

(2) A metric space (M,d) which is sequentially compact.

(3) A path-connected topological space that is not simply connected.

(4) A pair of topological spaces, say (X,τ_X) and (Y,τ_Y), and a function $f : X \to Y$ such that $f(\tau_X) \subset \tau_Y$ and f is not continuous.

(5) A topological space (X,τ) which is not locally compact.

(6) A topological space (X,τ) which is not locally path-connected.

(7) A topological space (X,τ) which is not first countable.

(8) A metric space (M,d) which is not complete.

(9) A metric space (M,d) and a subset $A \subset M$ which is nowhere dense and not closed.

(10) A metric space (M,d) and a subset $A \subset M$ which is dense and meagre.

Problem 3. *Let (M,d) be a metric space.*

(i) Assume that M with the metric topology is connected and that $|X| > 1$. Prove that M is uncountable.

(ii) Pick $A \subset M$ complete (w.r.t. the induced metric) and show that A is closed in M.

Problem 4. *Let X and Y be topological spaces, $f,g : X \to Y$ continuous maps, and assume that Y is Hausdorff.*

(i) Prove that $\{x \in X\,;\, f(x) = g(x)\}$ is closed in X.

(ii) Further assume that f is injective and X is compact. Prove that X is homeomorphic to $f(X)$.

Problem 5. *Recall that X/A denotes the quotient space where the subspace $A \subset X$ is identified with a point. Let $X = [0,1]$ and $A = (a,b)$, where $0 < a < b < 1$. Prove that X/A is connected and compact. Is it Hausdorff?*

Problem 6. *In this exercise, we consider the usual topologies. Using the fact that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(S^n) = \{0\}$ for $n > 1$ prove that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.*