Worksheet #13

Topology I - point set topology

December 17, 2024

Problem 1. *Let (X,τ) be a topological space. Prove that the statements below about a subset $A \subset X$ are all equivalent.*

(i) *A is nowhere dense, i.e. $int(cl(A)) = \emptyset$.*

(ii) *For every $U \in \tau$ non-empty, there exists $V \in \tau$ non-empty such that $V \subset U$ and $V \cap A = \emptyset$.*

(iii) *$cl(A)$ is nowhere dense.*

(iv) *$int(X \backslash A) = X \backslash cl(A)$ is dense in X.*

Problem 2. *Let (X,τ) be a topological space. Prove that a closed subset of X is nowhere dense if and only if it has an empty interior.*

Problem 3. *Prove that a finite union of nowhere-dense sets is nowhere-dense.*

Problem 4. *Let (X,τ) be a topological space. Prove that the statements below about subsets of X are all equivalent.*

(i) *The complement of every meagre subset is dense.*

(ii) *The interior of every meagre subset is empty.*

(iii) *The empty set is the only subset that is both open and meagre.*

(iv) *A countable intersection of dense open subsets is dense.*

(v) *A countable union of nowhere-dense closed subsets has no interior point.*

Problem 5. *Prove that a closed proper subspace of a normed vector space is always nowhere dense.*

Problem 6. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{n\to\infty} f(nx) = 0$ for all $x \in [0,\infty)$. Prove that $\lim_{x\to\infty} f(x) = 0$.*