Worksheet #12

Topology I - point set topology

December 10, 2024

Problem 1. *Let (M, d) be a metric space and let (X, τ) be a topological space.*

(i) Assume that $C(X, M) = \{f : X \to M \ f \text{ is continuous}\}$ is complete with the truncated sup metric (see, e.g., Section 4.5 in the notes). Prove that (M, d) must also be complete.

(ii) Assume that (X, τ) is compact. Prove that the sup metric and the truncated sup metric on the space $C(X, M)$ are topologically equivalent but not necessarily Lipschitz equivalent (Definition on page 59 of the notes).

Problem 2. *Prove that if $A \subset (C([0, 1], \mathbb{R}), d_\infty)$ is compact, then A is uniformly bounded and equicontinuous.*

Problem 3. *Consider the following sequences of continuous functions $[0, 1] \to \mathbb{R}$ and show that they do not have a convergent subsequence.*

(i) $f_n(x) = \sin(nx)$.

(ii) $g_n(x) = x^n$.

Problem 4. *Consider a sequence (f_n) of differentiable functions $[0, 1] \to \mathbb{R}$ satisfying $|f_n(x)| \le 1$ and $|f_n'(x)| \le 1$ for all $x \in [0, 1]$. Show that (f_n) has subsequential limits in the sup metric. Are these limits necessarily differentiable?*

Problem 5. *Let $p \ge 1$. Show that a subset $A \subset \ell^p$ is compact* [1] *if and only if A is closed, bounded and for every $\varepsilon > 0$ we can find some $N \in \mathbb{N}$ such that for all $x = (x_n) \in A$ we have that $\sum_{n \ge N} |x_n|^p < \varepsilon$.*

Problem 6. *Let (M, d) be a metric space. Consider the set S of all Cauchy sequences in (M, d). Define an equivalence relation on S by declaring that two Cauchy sequences (x_n) and (y_n) are equivalent iff $d(x_n, y_n) \to 0$ as $n \to \infty$. Denote by Z the set of all equivalence classes.*

(i) Given any two classes $[x], [y]$ in Z, choose representatives $(x_n) \in [x]$ and $(y_n) \in [y]$ and show that $d_Z([x], [y]) := \lim_{n\to\infty} d(x_n, y_n)$ defines a metric on Z.

(ii) Denote by $\tilde{M} \subset Z$ the set of equivalence classes of constant sequences. Define $\hat{M} := M \cup (Z \setminus \tilde{M})$ and construct a metric $\hat{d}$ on $\hat{M}$ whose restriction to M is given by d.

(iii) Prove that $(\hat{M}, \hat{d})$ is be complete and $\hat{M} = cl(M)$.

[1] in the topology induced by the metric we defined in Worksheet #11.