

Worksheet #11

Topology I - point set topology

Dec 3, 2024

Problem 1. Let (M, d) be a complete metric space. Prove that the connected components of (M, τ^d) are complete subspaces of (M, d) .

Problem 2. Consider the spaces of sequences $\ell^p \subseteq \mathbb{R}^{\mathbb{N}}$ for $p \geq 1$ as metric spaces. That is, for any $p \geq 1$ consider the set $\ell^p \subseteq \mathbb{R}^{\mathbb{N}}$ of all sequences (x_n) such that $\sum_{n \geq 1} |x_n|^p < \infty$ and endow this space with the metric

$$d_p(x, y) = \left(\sum_{n \geq 1} |x_n - y_n|^p \right)^{1/p}.$$

(i) Prove that the spaces ℓ^p are complete.

(ii) Show that the closed unit ball of ℓ^p around the zero sequence is not compact for any $p \geq 1$.

(iii) Now, consider the closed unit ball of ℓ^p around the zero sequence as a subset of ℓ^q with $q > p$, and prove that such ball is also not compact in the ℓ^q topology.

Problem 3. Let $f : (M, d_M) \rightarrow (N, d_N)$ be a continuous map and assume that there exists a constant $C > 0$ such that $d_N(f(x), f(y)) \geq C \cdot d_M(x, y)$ for all $x, y \in M$. Assume further that (M, d_M) is complete and prove that f is a closed map, meaning that if $A \subset M$ is closed, then $f(A)$ is closed in N .

Problem 4. Let (M, d_M) be a compact metric space (i.e., (M, τ^{d_M}) is compact) and let (N, d_N) be a complete metric space. Let $C(M, N) := \{f : M \rightarrow N ; f \text{ is continuous}\}$.

(i) Prove that $d(f, g) := \sup_{p \in M} d_N(f(p), g(p))$ defines a metric on $C(M, N)$.

(ii) Prove that $C(M, N)$ with this distance d is a complete metric space.

Problem 5. Consider $\mathbb{R}^N$ and $\mathbb{R}$ with their usual topologies/metrics. Let $K_1 \subset K_2 \subset \dots$ be a sequence of compact subsets of $\mathbb{R}^N$ such that $\bigcup_{n \in \mathbb{N}} K_n = \mathbb{R}^N$ and for all $n \in \mathbb{N}$ we have that $K_n \subset \text{int}(K_{n+1})$. Denote by $\|\cdot\|$ the norm on $C(K^n, \mathbb{R})$ induced by the usual sup metric d as in the previous problem.

(i) Prove that the function $\tilde{d}(f, g) := \sum_{n \in \mathbb{N}} \frac{2^{-n} \|(f - g)|_{K^n}\|}{1 + \|(f - g)|_{K^n}\|}$ defines a metric on the space $C(\mathbb{R}^N, \mathbb{R})$ of all continuous functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$.

(ii) Prove that $C(\mathbb{R}^N, \mathbb{R})$ with this distance $\tilde{d}$ is a complete metric space.

Problem 6 (Banach fixed point theorem). Let (M, d) be a complete metric space and $f : (M, d) \rightarrow (M, d)$ a map for which we can find some constant $C < 1$ such that $d(f(x), f(y)) \leq C \cdot d(x, y)$ for all $x, y \in M$. Prove that f has a unique fixed point.