Worksheet #10

Topology I - point set topology

November 26, 2024

To solve some of the exercises, you will need the definition below.

Definition 1. *If (M, d_M) and (N, d_N) are metric spaces, a function $\varphi : M \to N$ is called an isometric immersion if $d_M(p, q) = d_N(\varphi(p), \varphi(q))$ for all $p, q \in M$. We further say that φ is an isometry if φ is surjective.*

Problem 1. *Let $(G, +)$ be an Abelian group and assume that there exists a function $\varphi : G \to \mathbb{R}$ such that*

(a) $\varphi(0) = 0$ and $\varphi(g) > 0$ whenever $g \neq 0$,

(b) $\varphi(-g) = \varphi(g)$ for all $g \in G$ and

(c) $\varphi(g + h) \leq \varphi(g) + \varphi(h)$ for all $g, h \in G$

Prove that the function $d : G \times G \to \mathbb{R}$ given by $d(g, h) \coloneqq \varphi(g - h)$ is a metric on G.

Problem 2. *Let (M, d) be a metric space and consider $M \times M$ with the natural metric d_π induced by d. Choose $p \in M \times M$ such that $p \notin \Delta \coloneqq \{(x, x) \in M \times M\}$. Prove that $d_\pi(p, \Delta) > 0$.*

Problem 3. *Let d_1 and d_2 be equivalent metrics on a set M. Prove that the functions d_+ and d_{max} given by $d_+(p, q) \coloneqq d_1(p, q) + d_2(p, q)$ and $d_{max}(p, q) \coloneqq \max\{d_1(p, q), d_2(p, q)\}$ are metrics on M which are equivalent to both d_1 and d_2.*

Problem 4. *Let $M = \mathbb{R}\backslash\{-1, 1\}$ and consider M as a metric space, with the metric d induced from the Euclidean metric on $\mathbb{R}$. Prove that $\{x \in M; d(0, x) \leq 1\}$ is open in M.*

Problem 5. *Let $X \subset \mathbb{R}^2$ and assume that the Euclidean metric on $\mathbb{R}^2$ induces the discrete metric on X. Prove that $|X| \leq 3$.*

Problem 6. *Let (M, d_M) and (N, d_N) be two metric spaces and $\varphi : M \to N$ an isometric immersion. Prove that φ is injective.*

Problem 7. *Let (N, d_N) be a metric space and let M be a set. If $\varphi : M \to N$ is any injection, prove that $d_M(p, q) \coloneqq d_N(\varphi(p), \varphi(q))$ is the unique metric on M which makes φ into an isometric immersion.*

Problem 8. *Let V be a finite-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$. Let $T : V \to V$ be a linear transformation. Prove that the following statements are all equivalent.*

(i) $|T(v)| = |v|$ for all $v \in V$ (the norm $|\cdot|$ is the one induced from the inner product).

(ii) $\langle T(u), T(v) \rangle = \langle u, v \rangle$ for all $u, v \in V$.

(iii) T is an isometry (the metric is the one induced from $|\cdot|$).

Bonus problem

Problem 9. *Prove that any bounded metric space (M, d) can be isometrically immersed into the normed vector space $B(M, \mathbb{R}) = \{f : M \to \mathbb{R}\,;\, f \text{ is bounded}\}$, where on $B(M, \mathbb{R})$ one considers the metric (coming from the norm) given by $|f - g| \coloneqq \sup_{p \in M} |f(p) - g(p)|$.*