

Week 14 - Review session

Topology I - point set topology

December 18, 2024

Problem 1. Give an example of each of the following.

- (a) A topological space (X, τ) such that every subset $A \subset X$ is both open and closed.
- (b) A topological space (X, τ) and a subset $A \subset X$ which is neither open nor closed.
- (c) An open map $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$, which is not continuous.
- (d) A continuous bijection between two topological spaces, which is not a homeomorphism.
- (e) A simply connected topological space (X, τ) and connected subsets $U, V \in \tau$ such that $U \cup V$ is connected but not $U \cap V$.
- (f) A complete metric space (M, d) which is not compact (w.r.t. the metric topology).
- (g) A metric space (M, d) and a subset $A \subset M$ which is both dense and meagre.
- (h) A topological space (X, τ) which is not metrizable.
- (i) A metric space (M, d) which is not a Baire space.

Problem 2. Let X be a set with at least two elements and endow it with the indiscrete topology. Pick two distinct points $p, q \in X$ and consider a sequence (x_n) in X given by $x_n = p$ if n is even, and $x_n = q$ otherwise. Does this sequence converge?

Problem 3. True or false?

- (a) If (X, τ) is a topological space which is connected, then we can always find a proper subset $\emptyset \neq A \subsetneq X$ with an empty boundary.
- (b) Let (X, τ) be a topological space, $A \subset X$ and denote by τ_A the corresponding subspace topology. Then there exists $U \subset A$ such that $U \in \tau$ but $U \notin \tau_A$.
- (c) If (X, τ) is path connected, then every continuous function $f : (X, \tau) \rightarrow (\{0, 1\}, \tau_{\text{disc}})$ is constant.
- (d) Any set endowed with the discrete metric is complete.
- (e) Any set endowed with the cofinite topology is compact.
- (f) If (M, d) and (N, d') are two metric spaces such that (M, τ^d) and $(M, \tau^{d'})$ are homeomorphic, then M is complete if and only if N is complete.
- (g) If (M, d) is a metric space and $\emptyset \neq A \subset M$ is nowhere dense, then $A \notin \tau^d$.

Problem 4. *Define the following concepts.*

- (a) Metric topology.*
- (b) Product topology.*
- (c) Connected topological space.*
- (d) Compact topological space.*
- (e) Hausdorff topological space.*
- (f) Normal topological space.*
- (g) First-countable topological space.*
- (h) The fundamental group of a path-connected topological space.*