Math 220 - Lecture #9

November 12, 2024

Today:
- Compactness for products
- Heine-Borel in $\mathbb{R}^n$
- One-point compactifications

Proposition (Prop. 3.19) Let $(X_1, \tau_1), \dots, (X_n, \tau_n)$ be compact topological spaces. Then $X_1 \times \cdots \times X_n$ with the product topology is also compact.

proof It suffices to consider the case $n = 2$.

Consider

$$\beta = \{ \mathcal{U} \times V \, ; \, \mathcal{U} \in \tau_1 \text{ and } V \in \tau_2 \}$$

Then, τ^β is the product topology on $Y := X_1 \times X_2$ and it suffices to consider covers of Y of the form $\bigcup_{i \in I} (\mathcal{U}_i \times V_i)$ $\star$.

Pick $x_1 \in X_1$ and let $J = J(x_1) := \{ i \in I ; \, x_1 \in \mathcal{U}_i \}$

Since $\{x_1\} \times X_2 \subset \bigcup_{i \in J} (\mathcal{U}_i \times V_i)$, it follows that

$X_2 \subset \bigcup_{i\in J} V_i$. Now, because X_2 is

compact, $\exists\, J' \subset J$ finite, $J' = J'(x_1)$, such that

$X_2 \subset \bigcup_{i\in J'} V_i$ and $\{x_1\} \times X_2 \subset \bigcup_{i\in J'} (\mathcal{U}_i \times V_i)$.

Thus, letting $\mathcal{U}_{x_1} := \bigcap_{i\in J'} \mathcal{U}_i$ we have that

$\mathcal{U}_{x_1} \times X_2 \subset \bigcup_{i\in J'} (\mathcal{U}_i \times V_i)$.

But then, we can vary $x_1 \in X_1$ and consider

$X_1 = \bigcup_{x_1\in X_1} \mathcal{U}_{x_1}$. Since X_1 is also compact,

there exists some finite subset $A \subset X_1$ such that

$X_1 = \bigcup_{x_1\in A} \mathcal{U}_{x_1} = \bigcup_{i\in\tilde{J}} \mathcal{U}_i$, where

$\tilde{J} := \{ i \in J'(x_1) \, ; \; x_1 \in A \}$ is finite.

Thus, $$Y = X_1 \times X_2 = \bigcup_{i\in\tilde{J}} (\mathcal{U}_i \times V_i),$$

i.e., we have exihibited a finite subcover of $\star$.

∎

A few words on infinite products:

Let I be an <u>infinite</u> index set and consider a collection $\{(X_i, \tau_i)\}_{i\in I}$ of topological spaces. The product topology on $Y := \prod_{i\in I} X_i$ is defined

to be the topology generated by the collection of elements of the form

$$\prod_{i\in I} \mathcal{U}_i \quad \text{such that } \mathcal{U}_i \in \tau_i \ \underline{\text{AND}}$$

$\mathcal{U}_i \neq X_i$ only for finitely many X_i.

- This is the smallest topology on Y making all the projections continuous.

- Assuming the Axiom of choice, the previous proposition still holds for infinite products. This is Tychnoff's theorem.

Thm 3.20

Theorem (Heine-Borel) Consider $\mathbb{R}^n$ with the usual Euclidean topology. Then $K \subset \mathbb{R}^n$ is compact if and only if K is closed and <u>bounded</u> (i.e., contained in a ball).

Lemma Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $A \subset X$, $B \subset Y$.

Consider

① τ_1 = subspace topology on $A \times B \subset X \times Y$, where on $X \times Y$ we consider the product topology;

AND

② τ_2 = product topology on $A \times B$, where on $A \subset X$ and $B \subset Y$ we consider the subspace topology.

Then $\tau_1 = \tau_2 = \tau^{\beta}$, where

$$\beta := \{ (U \cap A) \times (V \cap B) \ ; \ U \in \tau_X \ \& \ V \in \tau_Y \}.$$

proof (of theorem)

($\Rightarrow$) If $K \subset \mathbb{R}^n$ is compact, then K is closed, since $(\mathbb{R}^n, \tau_{usual})$ is Hausdorff (lecture #8).

Moreover, since

$$d := d(\_, \{\underline{0}\}) : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x \longmapsto d(x, \underline{0})$$

is continuous, $d|_K$ is bounded. Thus, $K \subset B(0, R)$ for some $R > 0$.

($\Leftarrow$) Since K is bounded, $K \subset [-a, a]^n$ for some $a > 0$. Since $[-a, a]^n$ is compact and K is closed (also in $[-a, a]^n$), K is compact (as a subspace of $[-a, a]^n$, hence as a subspace of $\mathbb{R}^n$). $\blacksquare$

Corollary In $(\mathbb{R}^n, \tau_{usual})$, all closed balls are compact.

One-point compactifications

Definition A compactification of a topological space (X, τ) is an embedding of X as a dense subspace of a compact topological space.

This means that we have some $f : (X, \tau) \to (Y, \tilde{\tau})$ such that $f : X \to f(X)$ is a homeomorphism & $\overline{f(X)} = Y$, where $(Y, \tilde{\tau})$ is compact.

Definition Let (X, τ) be a topological space and ∞ a symbol that is not an element of X. Define $X^* := X \cup \{\infty\}$ and

$$\tau^* := \tau \cup \{ (X \setminus K) \cup \{\infty\} ; \ K \text{ closed and compact} \}$$

Proposition If (X, τ) is not compact then

① (X^*, τ^*) is compact and

② The natural inclusion $i : X \hookrightarrow X^*$ is an embedding with dense image.

▶ (X^*, τ^*) is thus called the one-point compactification of (X, τ).
(or Alexandroff compactification)

Question: When is this space Hausdorff?

Definition A topological space (X, τ) is called locally compact iff given $x \in X$ we can find $U \in \tau$ and $K \subset X$ compact such that $x \in U \subset K$.

Proposition A topological space (X, τ) is locally compact AND Hausdorff if and only if (X^*, τ^*) is Hausdorff.

Examples

- $(\mathbb{R}^n, \tau_{usual})$ is locally compact & Hausdorff
- $(X, \tau_{discrete})$ is " & "