A quick recap of lecture #7

(X,τ) a path-connected top. space, $p \in X$

- $\pi_1(X,p) := \{\gamma : [0,1] \to X \,;\, \gamma \text{ is a loop @ } p\} / \sim$

  is a group & independent of p

- (X,τ) is simply-connected $\iff \pi_1(X) = \{e\}$

- The circle is <u>not</u> simply-connected:

$$\pi_1(S^1) \simeq \mathbb{Z}$$

$$[\gamma] \mapsto \deg(\gamma) := \tilde{\gamma}(1) - \tilde{\gamma}(0)$$

lifted path to the helix $\simeq \mathbb{R}$

'=' # of times we "wrap" around the circle

Math 220 - Lecture # 8

November 5, 2024

Today: Compactness

Definition (Def 3.1) A topological space (X, τ) is called **COMPACT** if any open cover of X admits a finite subcover. A subset $K \subset X$ is compact iff it is compact with the subspace topology.

Important: We will use the following facts.

① Given (X, τ) and $K \subset X$, K is compact $\iff$ every cover of K by opens of X admits a finite subcover.

② In particular, if $\tau = \tau^{\mathcal{B}}$ it suffices to look at covers by sets of $\mathcal{B}$.

Examples

(A) Any set X with the indiscrete topology $\tau = \{X, \phi\}$.

(B) Any finite set X with the discrete topology $\tau = \mathcal{P}(X)$.

(C) Any closed interval $[a,b] \subset \mathbb{R}$ (usual topology). (c.f. Prop. 3.2)

Write $[a,b] \subset \bigcup_{i \in I} \mathcal{U}_i$, $\mathcal{U}_i \subset \mathbb{R}$ open.

Let $Y = \{ y \in [a,b] ; [a,y]$ admits a finite subcover of $\{\mathcal{U}_i\} \}$.

We want to show that $s := \sup Y \in Y$ & $s = b$

(note that $a \in Y \Rightarrow Y \neq \phi \Rightarrow \sup Y$ exists & $\sup Y \leq b$ as b is an upper-bound)

(Note that (as already observed) a can be covered by a single open $\mathcal{U}_j$ & $\exists$ some $\varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subset \mathcal{U}_j$ (because $\mathcal{U}_j$ is open). Thus, $s > a$.

$\Downarrow$

$[a, a+\varepsilon) \subset \mathcal{U}_j \Rightarrow a+\varepsilon/2 \in Y$)

Since $s \in [a,b]$, $\exists\, \mathcal{U}_j$ such that $s \in \mathcal{U}_j$

Moreover, because $\mathcal{U}_j$ is open, $\exists\, \delta > 0$ such that

$$(s-\delta, s+\delta) \subset \mathcal{U}_j$$

Now, $s-\delta$ is not an upper bound for the set Y.

Thus, up to taking a smaller δ, we may further assume that $a < s - \delta/2$ (see $\uparrow$) and that $s - \delta/2 \in Y$. But then $\exists\, \mathcal{U}_{i_1}, \dots, \mathcal{U}_{i_n}$ covering $[a, s-\delta/2]$ and

$$[a,b] \cap [s-\delta/2, s+\delta) \subset \mathcal{U}_j \cup \mathcal{U}_{i_1} \cup \dots \cup \mathcal{U}_{i_n} \quad \bigstar$$

So, $s \in Y$ and it must be the case that $s = b$.

If $s < b$, then $\bigstar$ shows that $s + \delta' \in Y$ for some $\delta' < \delta$. But this is not possible as $s = \sup Y$. ∎

(D) Any set $X \neq \emptyset$ with

$$\tau = \{ A \subset X ;\ x \notin A \text{ or } X \setminus A \text{ finite} \},$$

where $x \in X$ is any fixed point.

Write $X = \bigcup_{i \in I} \mathcal{U}_i$ with $\mathcal{U}_i \in \tau$.

Then $\exists\, j \in I$ s.t. $X \setminus \mathcal{U}_j$ is finite (could be $= \emptyset$)

and $x \in \mathcal{U}_j$.

Then either $X = \mathcal{U}_j$ and we are done, or

we can write $X = \mathcal{U}_j \cup \{x_1, \dots, x_n\}$

for some $n \in \mathbb{N}$. But then for each $k = 1, \dots, n$

$\exists\, \mathcal{U}_k$ with $x_k \in \mathcal{U}_k$ and

$$X = \mathcal{U}_j \cup \mathcal{U}_1 \cup \dots \cup \mathcal{U}_n .$$

Proposition (Prop. 3.8) If (X, τ_X) is a compact topological space and $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, then $f(X) \subset Y$ is compact.

proof Consider $\bigcup_{i \in I} \mathcal{U}_i \supset f(X)$ with $\mathcal{U}_i \in \tau_Y$.

Let $V_i := f^{-1}(\mathcal{U}_i)$. Since f is continuous, $V_i \in \tau_X$. Now, $X = \bigcup_{i \in I} V_i$ and since X is compact, $\exists\ I' \subset I$ finite such that $X = \bigcup_{i \in I'} V_i$.

But then $f(X) \subset \bigcup_{i \in I'} \mathcal{U}_i$.

$$\left(\text{Note: } f\left(\bigcup V_i\right) = f\left(\bigcup f^{-1}(\mathcal{U}_i)\right) = f\left(f^{-1}\left(\bigcup \mathcal{U}_i\right)\right) \subset \bigcup \mathcal{U}_i \right)$$

$\boxtimes$

Corollary If (X, τ_X) and (Y, τ_Y) are homeomorphic, then (X, τ_X) is compact $\Leftrightarrow$ (Y, τ_Y) is compact

Examples of how to apply this (usual topologies)

① $\underbrace{\mathbb{R} \text{ is not compact}}$ $\Rightarrow$ $\mathbb{R}$ is not homeomorphic to $[a,b]$

The cover by the opens

$\mathcal{U}_n = (-n, n)$ for $n \in \mathbb{N}$

does not admit a finite

subcover.

② Any open interval $(a,b) \subset \mathbb{R}$ is not compact since $(a,b) \underset{\text{homeo}}{\sim} \mathbb{R}$.

In fact, in Analysis you learn that for $(\mathbb{R}^n, \tau_{usual})$

- $K \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ K is closed and bounded

When (X, τ) is a compact top. space, the implication $K \subset X$ closed $\Rightarrow$ $K \subset X$ compact is always true. HOWEVER, the reverse one fails in general.

For example, X infinite, τ cofinite & $K \subset X$ infinite.

Proposition (Prop. 3.13) If (X, τ) is compact and $K \subset X$ is closed, then K is compact.

proof Write $K \subset \bigcup_{i \in I} \mathcal{U}_i$ with $\mathcal{U}_i \in \tau$.

Since K is closed, $X \setminus K$ is open and $(X \setminus K) \cup \left(\bigcup_{i \in I} \mathcal{U}_i \right)$ is an open cover of X.

But X is compact, so $\exists$ $I' \subset I$ finite s.t.

$$X = (X \setminus K) \cup \left(\bigcup_{i \in I'} \mathcal{U}_i \right)$$

and hence $K \subset \bigcup_{i \in I'} \mathcal{U}_i$.

$\blacksquare$

The converse is true if we assume that (X, τ) is Hausdorff.

Recall (Worksheet #4) (X,τ) is Hausdorff iff given $x, y \in X$ $(x \neq y)$, $\exists\, \mathcal{U}_x, \mathcal{U}_y \in \tau$ disjoint such that $x \in \mathcal{U}_x$ and $y \in \mathcal{U}_y$.

Proposition (Prop. 3.K) If (X,τ) is Hausdorff, then every compact subset of X is closed.

proof Pick $K \subset X$ compact. We want to show that any $x \in X \setminus K$ is an interior point (of $X \setminus K$, i.e. $X \setminus K$ is open).

This means that we want to find $\mathcal{U}_x \in \mathcal{T}$ such that $x \in \mathcal{U}_x$ and $\mathcal{U}_x \subset X \setminus K$.

So, fix $x \in X \setminus K$. Since X is Hausdorff, given any $y \in K$ we can find disjoint opens $\mathcal{U}_{x,y}$ and $\mathcal{U}_y$ with $x \in \mathcal{U}_{x,y}$ & $y \in \mathcal{U}_y$

Now, $K \subset \bigcup_{y \in K} \mathcal{U}_y$ and since K is compact $\exists\, y_1, \dots, y_n \in K$ s.t. $K \subset \bigcup_{i=1}^{n} \mathcal{U}_{y_i}$.

But then $\mathcal{U}_x := \bigcap_{i=1}^{n} \mathcal{U}_{x,y_i}$ is open and, by construction, $x \in \mathcal{U}_x \subset X \setminus K$.

$\boxtimes$

Example Every metric space is Hausdorff. Thus, on a <u>compact</u> metric space (M, d),

$$K \subset M \text{ compact} \iff K \subset M \text{ is closed}$$

One example is given by any finite set M with the discrete metric.

A criterion for compacteness & Cantor's thm

Lemma (Lemma 3.11) A top. space (X, τ) is compact if and only if given any collection $\{C_j\}_{j \in J}$ of closed subsets with $\bigcap_{j \in J} C_j = \emptyset$, we can find $J' \subset J$ finite such that $\bigcap_{j \in J'} C_j = \emptyset$.

proof of $\Rightarrow$ If (X, τ) and $\{C_j\}$ are as above, then

$$X = X \setminus \emptyset = X \setminus \left(\bigcap_{j \in J} C_j \right) = \bigcup_{j \in J} \underbrace{(X \setminus C_j)}_{\in \tau}$$

But then, because (X, τ) is compact, $\exists\, J' \subset J$ finite such that

$$X = \bigcup_{j \in J'} (X \setminus C_j)$$

But then $\underset{\phi}{\overset{\shortparallel}{X^c}} = \bigcap_{j \in J'} (X \setminus C_j)^c = \bigcap_{j \in J'} C_j$.

$\boxtimes$

Corollary (Cantor) $\rightsquigarrow$ This is Cor. 3.12

If (X, τ) is compact and $\{C_n\}_{n \in \mathbb{N}}$ is a sequence of closed non-empty subsets with

$$C_n \supset C_{n+1} \supset C_{n+2} \cdots \quad \forall n,$$

then $\bigcap_{n \in \mathbb{N}} C_n \neq \phi$

proof By contradiction, assume that $\bigcap_{n \in \mathbb{N}} C_n = \phi$.

Then, by the previous lemma, $\exists\, n_1, \dots, n_k$ s.t. $\bigcap_{i=1}^{k} C_{n_i} = \phi$. But then (up to relabeling) we may assume that $n_1 < n_2 < \dots < n_k$ and by assumption we have that

$$C_{n_1} \supset C_{n_2} \supset \cdots \supset C_{n_K}$$

hence

$$\emptyset = \bigcap_{i=1}^{K} C_{n_i} = C_{n_K},$$

which contradicts the fact that $C_{n_K} \neq \emptyset$. (each C_j is non-empty) $\blacksquare$

Alternative proof that $[a,b]$ is compact:

By contradiction, assume that $\exists \{\mathcal{U}_i\}$ s.t. $[a,b] \subset \bigcup \mathcal{U}_i$, $\mathcal{U}_i \subset \mathbb{R}$ open but $\nexists$ finite subcover. Divide the interval in half:

$$[a,b] = [a, a + b-a/2] \cup [a+b-a/2, b]$$

Then at least one of the halves cannot be covered by finitely many of the $\mathcal{U}_i$ s. Divide this interval in half and proceed this way, inductively, to construct @ the n-th step a closed interval I_n of length $b-a/2^n$ that admits no finite sub-cover of $\{\mathcal{U}_i\}$ and such that

$$I_1 \supset I_2 \supset \cdots \supset I_n$$

Writing $I_n = [a_n, b_n]$, and letting $A = \sup a_n$ and $B = \inf b_n$, it follows that $\bigcap_n I_n = \{x \in \mathbb{R};\ A \leq x \leq B\} \neq \phi$.

Thus, let $p \in \bigcap_n I_n$. Since $p \in [a,b]$, $\exists\, \mathcal{U}_j$ s.t. $p \in \mathcal{U}_j$ and since $\mathcal{U}_j$ is open $\exists\, \delta > 0$ s.t. $(p-\delta, p+\delta) \subset \mathcal{U}_j$. But then, by taking n large enough, $I_n \subset (p-\delta, p+\delta)$ and $I_n \subset \mathcal{U}_j$, which is a contradiction.

$\blacksquare$