Math 220 - Lecture #6

October 15, 2024

Today: More on connectedness
+
intro to simple-connectedness

Definition Let (X, τ) be a topological space. We say that X is locally path-connected iff given $x \in X$ and $V \in \tau$ with $x \in V$, we can find $\mathcal{U} \in \tau$ such that $x \in \mathcal{U}$, $\mathcal{U} \subset V$ and $\mathcal{U}$ is path-connected.

Examples

① $(\mathbb{R}^n, \tau_{\text{usual}})$ because all open balls are convex, in particular, path-connected.

② Every open subset of a locally path-connected space.

(X, τ) locally path-connected

$A \subset X$ open, τ_A = subspace topology

$x \in A$, $V \in \tau_A$ with $x \in V$

$V = A \cap \tilde{V}$ for some $\tilde{V} \in \tau$.

We need to show that $\exists\ \mathcal{U} \in \tau_A$

path-connected with $x \in \mathcal{U} \subset V$.

Since A is open in X, $V \in \tau$.

Since X is locally path-connected $\exists$

$\tilde{\mathcal{U}} \in \tau$ path-connected s.t. $x \in \tilde{\mathcal{U}} \subset V$.

But $\tilde{\mathcal{U}} = \tilde{\mathcal{U}} \cap A$ (since $V \subset A$)

and we can take $\mathcal{U} = \tilde{\mathcal{U}} \in \tau_A$.

$\blacksquare$

Proposition Let (X, τ) be a locally path-connected topological space. Then X is connected iff X is path-connected.

Corollary (Theorem 2.12) Given $A \subset \mathbb{R}^n$ open, we have that A is connected iff A is path-connected (by 1 + 2).

proof of proposition

($\Leftarrow$) always true (Lecture #5)

($\Rightarrow$) Assume that X is connected and write $x \sim y \iff \exists$ a path (in X) from x to y.
Choose $x \in X$ and let $A = \{ y \in X ; y \sim x \}$.
Since X is locally path-connected, both A and $X \setminus A$ are open. Since $A \neq \emptyset$ as $x \in A$,
($\iff$ A is closed)
and X is connected we must have that $X \setminus A = \emptyset$ and $A = X$. ∎

Why are A and $X \setminus A$ open?

Pick $a \in A$ we want to show that $a \in \text{int}(A)$.
($A \neq \emptyset$)

Since X is locally path-connected and $X \in \tau$, $\exists\, \mathcal{U} \in \tau$ path-connected with $a \in \mathcal{U}$.

But then $b \in \mathcal{U} \Rightarrow b \sim a$ and since $a \sim x$ we have that $b \sim x$. That is, $b \in A$ and $a \in \mathcal{U} \subset A \left(\Rightarrow a \in \operatorname{int}(A) \right)$.

Similarly, if $X \setminus A \neq \emptyset$, then picking $c \in X \setminus A$, $\exists\, V \in \tau$ path-connected with $c \in V$.

But then $d \in V \Rightarrow d \sim c \Rightarrow d \not\sim x$ since $c \not\sim x$

$\Rightarrow d \in X \setminus A$.

That is, $c \in V \subset X \setminus A$.

Unions, products & closure

Proposition Let (X, τ) be a topological space. Consider a family $\{A_\lambda\}$ of connected subsets of X. If $\bigcap A_\lambda \neq \emptyset$, then $Y := \bigcup A_\lambda$ is connected.

proof

By contradiction, assume that $Y = (\mathcal{U} \cup V) \cap Y$ with $\mathcal{U}, V \in \mathcal{T}$ such that $\mathcal{U} \cap Y \neq \phi$, $V \cap Y \neq \phi$ and $\mathcal{U} \cap V \cap Y = \phi$.

Let $x \in \bigcap A_\lambda$. Then either

- $x \in \mathcal{U} \cap Y$ but $x \notin V \cap Y$, or
- $x \in V \cap Y$ but $x \notin \mathcal{U} \cap Y$.

W.L.O.G assume that the first holds.

Since $x \in \mathcal{U} \cap Y$, $\forall \lambda$ we have that $\mathcal{U} \cap A_\lambda \neq \phi$

Moreover, $\forall \lambda$ we can write disjoint

$$A_\lambda = \underbrace{(\mathcal{U} \cap A_\lambda)}_{\neq \phi} \,\dot{\cup}\, (V \cap A_\lambda) \qquad \bigstar$$

Now, because $V \cap Y = \bigcup (V \cap A_\lambda) \neq \phi$, there exists some λ_0 such that $V \cap A_{\lambda_0} \neq \phi$.

But then $\bigstar$ tells us that A_{λ_0} is not connected.

$\boxtimes$

Corollary A topological space (X, τ) is connected iff any two points $p, q \in X$ belong to some connected subspace $Y_{pq} \subset X$.

proof

$(\Rightarrow)$ given $p, q \in X$, simply take $Y_{pq} = X$.

$(\Leftarrow)$ Fix $p \in X$. Then we can write

$$X = \bigcup_{q \in X} Y_{pq} \text{ . Since } \{p\} \subset \bigcap Y_{pq},$$

it follows that X is connected.

Proposition Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Then $(X \times Y, \tau_{prod})$ is connected iff (X, τ_X) and (Y, τ_Y) are <u>both</u> connected.

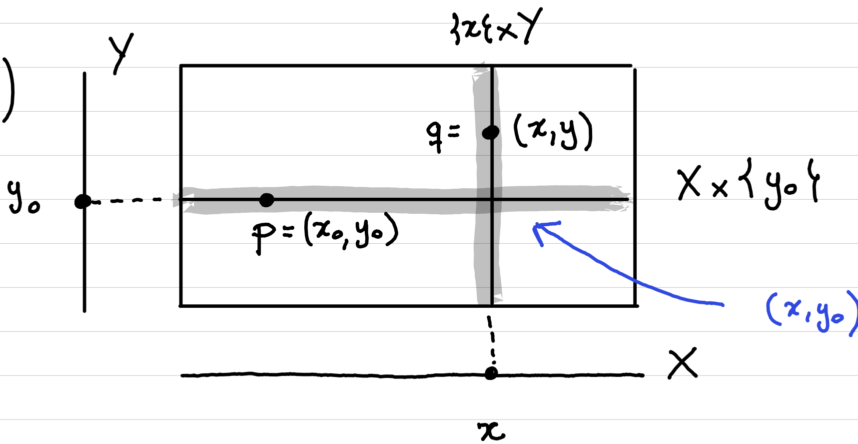
proof

($\Rightarrow$) If $(X \times Y, \tau_{prod})$ is connected, then looking at the continuous and surjective functions

$$\pi_X : X \times Y \twoheadrightarrow X \quad \text{and} \quad \pi_Y : X \times Y \twoheadrightarrow Y$$

we deduce that both X and Y are connected.

($\Leftarrow$)



Pick $p = (x_0, y_0) \in X \times Y$. Given $\overset{q}{\overset{\shortparallel}{(x,y)}} \in X \times Y$, we have that $A_q := (X \times \{y_0\}) \cup (\{x\} \times Y) \subset X \times Y$ is connected:

- $X \times \{y_0\}$ is connected since $\sim_{homeo} X$
- $\{x\} \times Y$ " since $\sim_{homeo} Y$
- $(X \times \{y_0\}) \cap (\{x\} \times Y) = \{(x, y_0)\}$

Moreover, $p \in A_q \ \forall \ q \in X \times Y$ and

$$X \times Y = \bigcup_{q \in X \times Y} A_q .$$ Thus, $X \times Y$ is connected

by the previous proposition (page 4).

$\boxtimes$

Proposition Let (X, τ) be a topological space. If $A \subset X$ is connected, then $cl(A)$ is also connected.

proof

By contradiction, assume that $\exists f : cl(A) \twoheadrightarrow \{0,1\}$ continuous.

Since A is connected and $A \subset cl(A)$, $f|_A$ must be constant. W.L.O.G, assume that $f|_A \equiv 1$.

Since $cl(A) = A \cup \partial A$ and f is surjective, we can pick $x \in \partial A$ such that $f(x) = 0$.

Now, since f is continuous, $\exists\, \mathcal{U}$ open in $cl(A)$ with $x \in \mathcal{U}$ such that $f(\mathcal{U}) \subset \{0\}$.

Write $\mathcal{U} = V \cap cl(A)$ with $V \in \mathcal{T}$.

Since $x \in \partial A$, $(x \in \mathcal{U} \Rightarrow)\ x \in V$ implies that $V \cap A \neq \phi$. But $V \cap A = \mathcal{U} \cap A$ and this contradicts the fact that $f|_A = 1$ and $f|_{\mathcal{U}} = 0$.

$\blacksquare$

Section 2.2 (Simple-connectedness)

Definition (2.24) Let $(X, \mathcal{T})$ be a topological space, $x, y \in X$, and $\gamma_0, \gamma_1 : [0,1] \longrightarrow X$ two paths from x to y. We say that γ_0 and γ_1 are path-homotopic, and write $\gamma_0 \cong \gamma_1$ iff we can find $H : [0,1] \times [0,1] \longrightarrow X$

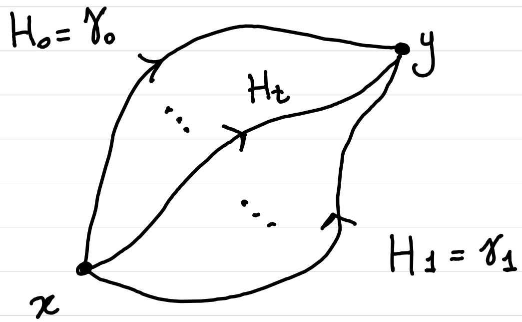
$$(s,t) \longmapsto H(s,t)$$

continuous such that

(i) $H(s,0) = \gamma_0(s)$ and $H(s,1) = \gamma_1(s)$ $\forall s \in [0,1]$,

(ii) $H(0,t) = x = \gamma_0(0) = \gamma_1(0)$ $\forall t \in [0,1]$, and

(iii) $H(1,t) = y = \gamma_0(1) = \gamma_1(1)$ $\forall t \in [0,1]$.



Here is a picture.

- In other words, we can think of H as a collection of paths
$$H_t : [0,1] \longrightarrow X$$
$$s \longmapsto H(s,t)$$
from x to y that "move" with t, "starting" (at $t=0$) from $H_0 = \gamma_0$ and "ending" (at $t=1$) with $H_1 = \gamma_1$.

- H (or $\{H_t\}$) is called a path-homotopy from γ_0 to γ_1.

Example

Consider $(X, \tau) = (\mathbb{R}^n, \tau_{usual})$. Then any two paths with the same endpoints are path-homotopic.

In fact,

$$H_t(s) = (1-t)\,\gamma_0(s) + t\,\gamma_1(s)$$

defines a path-homotopy from γ_0 to γ_1.