Math 220 - Lecture #5

October 8, 2024

Today: Connectedness

Proposition 1 Consider $(X, \tau) = (\mathbb{R}, \tau_{usual})$.
It is impossible to write $\mathbb{R} = A \cup B$ with $A \cap B = \emptyset$ and A and B both non-empty and open.

proof By contradiction, assume that we can find such $A, B \subset \mathbb{R}$. Take $a \in A$ and $b \in B$.
Since $A \cap B = \emptyset$, $a \neq b$ and we may assume that $a < b$. Let $X := \{x \in A ; x < b\} = A \cap (-\infty, b)$.
Then $a \in X$, $X \neq \emptyset$ and there exists $c = \sup X$. Moreover, $c \leq b$.

Now, by the definition of sup, given $\varepsilon > 0$, we can find $x \in X$ such that $c - \varepsilon < x \leq c$.

But since $X \subset A$, this $x \in X$ is an element of A. Hence, $c \in cl(A)$ since, varying $\varepsilon > 0$, this shows that every open interval centered at c intersects A.

Observe now that $A = X \setminus B$ is closed as B is open. Thus, $c \in A \; (= cl(A))$ and $c \neq b$, which further implies that $c < b$ and $c \in X$.

Next, since $X = A \cap (-\infty, b)$ is open, we Know that (definition of the metric topology) $\exists \, \delta > 0$ such that

$$(c - \delta, c + \delta) \subset X$$

And this contradicts the fact that $c = \sup X$.

$\boxtimes$

RmK.: The proof of Prop 1 can be adapted to show all intervals in $\mathbb{R}$ are connected.

See Prop 2.2 in the notes.

Definition A topological space (X, τ) is called **disconnected** iff we can find $U, V \in \tau$ disjoint, and both non-empty, such that

$$X = U \dot{\bigcup} V$$

Otherwise, (X, τ) is **connected**.

So, we have proved that $(\mathbb{R}, \tau_{usual})$ is an example of connected topological space.

Definition Given a topological space (X, τ), we say that $Y \subset X$ is connected if (Y, τ_{sub}) is connected.

subspace topology

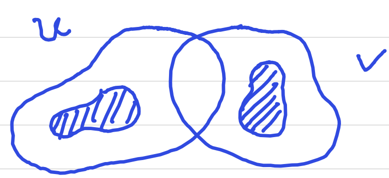
Note: This means that Y is not connected iff

$$Y = (U \cap Y) \dot{\bigcup} (V \cap Y) \quad \text{for some } U, V \in \tau \text{ with } U \cap Y \neq \emptyset,\ V \cap Y \neq \emptyset$$

$$= (U \cup V) \cap Y$$

$\Leftrightarrow$ $Y \subset U \cup V$, $U \cap Y \neq \emptyset$, $V \cap Y \neq \emptyset$ and $U \cap V \cap Y = \emptyset$

for some $U, V \in \tau$



//// Y

In particular, if you can find $\mathcal{U}, V \in \tau$ with $\mathcal{U} \cap V = \emptyset$, $\mathcal{U} \cap Y \neq \emptyset$, $V \cap Y \neq \emptyset$ and $Y \subset \mathcal{U} \cup V$, then you can conclude that Y is disconnected.

What is the contrapositive?

However, for the other implication, you really deduce that $\mathcal{U} \cap V \cap Y = \emptyset$ and not that $\mathcal{U} \cap V = \emptyset$.

Here is a counter-example:

- $X = \{1, 2, 3\}$ and $\tau = \{X, \emptyset, \{1,2\}, \{2,3\}, \{2\}\}$

- $Y = \{1, 3\}$

  $Y \subset X$ is disconnected because

  $Y = (Y \cap \{1,2\}) \dot\cup (Y \cap \{2,3\}) \quad \star$

  But $\mathcal{U} = \{1,2\}$ and $V = \{2,3\}$ are not disjoint. Note that $\star$ is the unique disconnection of Y.

Fist properties (Proposition 2)

① If $X = \mathcal{U} \bigcup V$ (as above) is disconnected, then $\mathcal{U}$ and V are both open and closed.

In fact, X is connected $\Leftrightarrow$ the only subsets of X which are both open and closed are $\emptyset$ and X.

② (X, τ) is disconnected $\Leftrightarrow$ $\exists f : (X, \tau) \twoheadrightarrow (\{0,1\}, \tau_{disc})$ continuous

Or equivalently,

(X, τ) is connected $\Leftrightarrow$ every continuous function $f : X \to \{0, 1\}$ is constant.

③ If (X, τ_X) is connected and $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, then $f(X) \subset Y$ is connected.

Why?

proof

① It suffices to observe that $\mathcal{U} = X \setminus V$ and $V = X \setminus \mathcal{U}$. Since V is open, $\mathcal{U}$ is closed; and since $\mathcal{U}$ is open, V is closed.

② ($\Rightarrow$) If (X, τ) is disconnected, then we can write $X = \mathcal{U} \cup V$, with $\phi \neq \mathcal{U}, V \in \tau$, $\mathcal{U} \cap V = \phi$. We can thus define

$$f : X \longrightarrow \{0, 1\}$$

$$x \longmapsto \begin{cases} 0, & \text{if } x \in \mathcal{U} \\ 1, & \text{if } x \in V \end{cases}$$

Now, because $\mathcal{U}$ and V are non-empty, f is is surjective. Moreover, since a), b), c) and d) below hold, we conclude that f is continuous

a) $f^{-1}(\{0\}) = \mathcal{U} \in \tau$

c) $f^{-1}(\{0, 1\}) = X \in \tau$

b) $f^{-1}(\{1\}) = V \in \tau$

d) $f^{-1}(\phi) = \phi \in \tau$

($\Leftarrow$) Conversely, given $f: X \twoheadrightarrow \{0,1\}$ continuous, letting $\mathcal{U} = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ we have that $\mathcal{U} \cap V = \emptyset$ by construction and, moreover,

- f continuous $\Rightarrow \mathcal{U}, V \in \tau$
- f surjective $\Rightarrow \mathcal{U}, V \neq \emptyset$ and
  $$\mathcal{U} \cup V = f^{-1}(\{0,1\}) = X$$

③ By contradiction, assume that $f(X)$ is disconnected. Then, by ②, $\exists\, g: f(X) \twoheadrightarrow \{0,1\}$ continuous. and consider
$$h: X \twoheadrightarrow f(X), \quad x \mapsto f(x).$$
Then h is continuous,

hence $g \circ h: X \twoheadrightarrow \{0,1\}$ is continuous, which (by ②) contradicts the fact that X is connected.

$\blacksquare$

Connected components

Given a topological space (X, τ) we can define an equivalence relation as follows.

$$\forall\, x, y \in X \qquad x \sim y \iff \exists\, Y \subset X \text{ connected with } x, y \in Y$$

Definition / proposition The equivalence classes of the above relation are called the connected components of X.

Given $x \in X$ we have that

$$C_x := \{ y \in X \,;\, y \sim x \}$$

= union of all connected subsets of X that contain x.

proposition part

= largest connected subset of X which contains x.

Path-connectedness

Definition A path in a topological space (X, τ) is a continuous map

$$\gamma : \left([0,1], \tau_{usual} \right) \longrightarrow \left(X, \tau \right).$$

If $x = \gamma(0)$ and $y = \gamma(1)$, we further say that γ is a path from x to y.

Definition A topological space (X, τ) is called path-connected iff for any two points $x, y \in X$ there exists a path $\gamma : [0,1] \to X$ from x to y.

RmK.: On X, we can also consider an equivalence relation defined by

$x \sim y \iff$ there exists a path from x to y

the equivalence classes are called the path components of X.

Question: Is being connected the same as being path-connected?

Proposition 3 If a topological space (X, τ) is path-connected, then it is connected as well.

(path-connectedness $\Rightarrow$ connectedness)

$\neq$ ⚠

proof 1

<u>Claim</u> To deduce that (X, τ) is conneted, it suffices to show that any two points $x, y \in X$ are contained in a connected subspace.
(That is, X has exactly one conneted component)

So, let's show that this is the case. Pick $x, y \in X$.
Then, by assumption, there exists a path $\gamma : [0,1] \to X$ from x to y.

But then $\gamma([0,1]) \subset X$ is a connected subspace (Proposition 2, 3) that contains both $x = \gamma(0)$ and $y = \gamma(1)$.

$\boxtimes$

proof 2 By contradiction, assume that X is not connected. Then, we can find $f : X \twoheadrightarrow \{0,1\}$ continuous.

Pick $x \in f^{-1}(\{0\})$ and $y \in f^{-1}(\{1\})$.

By assumption, we can also find a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. But then $f \circ \gamma : [0,1] \longrightarrow \{0,1\}$ is surjective and continuous, which contradicts the fact that $[0,1] \subset \mathbb{R}$ is connected (Proposition 2.2 in the notes).

$\boxtimes$

What about the converse?

Example

Consider $f : (0, +\infty) \longrightarrow \mathbb{R}$

$$x \longmapsto \cos\left(\frac{1}{x}\right)$$

Then f is continuous and $\mathrm{graph}(f) \underset{\text{homeo}}{\sim} (0, +\infty)$ is connected.

Now, $cl\big(\mathrm{graph}(f)\big) = \mathrm{graph}(f) \bigcup \underbrace{\{0\} \times [-1, 1]}_{:= I}$

and for any subset $S \subset I$ we can show that $\mathrm{graph}(f) \cup S$ is also connected.

In particular, $Y = \mathrm{graph}(f) \cup \{(0,0)\}$ gives an example of a top. space which is connected but <u>NOT</u> path-connected.

(more on Moodle)