Math 220 - Lecture #4

October 1st, 2024

Today: Subspaces, products, disjoint unions and quotients.

Definition (Subspace topology)

Given a topological space (X, τ) and $S \subset X$ we define the subspace topology on S by declaring $\mathcal{U} \subset S$ open iff $\mathcal{U} = V \cap S$ for some $V \in \tau$.

Example Consider $(X, \tau) = (\mathbb{R}, \tau_{usual})$ and $S = [0,1]$. Then $\mathcal{U} = (0,1]$ is open on S (w.r.t. the subspace topology) since

$$\mathcal{U} = \underbrace{[0,1]}_{= S} \cap \underbrace{(0, 1+\varepsilon)}_{\in \tau_{usual}}$$

Properties characterizing the subspace topology:

Let (X, τ) be a top. space and fix $S \subset X$.

Let τ_S be the subspace topology on S.

Then τ_S is characterized by the following properties:

1. If $f: (X, \tau) \longrightarrow (Y, \tau_Y)$ is continuous, then $f|_S: (S, \tau_S) \longrightarrow (Y, \tau_Y)$ is continuous.

2. The inclusion $i_S: S \hookrightarrow X$ is a <u>topological</u> <u>embedding</u>. This means that i_S is a homeomorphism onto its image.
   $(S, \tau_S) \longrightarrow (X, \tau)$

3. The subspace topology on S is the **smallest** topology on S for which the inclusion is continuous.

④ The subspace topology on S is the ==unique== topology on S for which the following universal property holds:

- If (Y, τ_Y) is another top. space, then $f: Y \to S$ is continuous iff $i_S \circ f: Y \to X$ is continuous:

$(Y, \tau_Y) \to (S, \tau_S)$ $\qquad$ $(Y, \tau_Y) \to (X, \tau)$

$$\begin{tikzcd} S \arrow[r, hook, "i_S"] & X \\ Y \arrow[u, "f"] \arrow[ur, "i_S \circ f"'] & \end{tikzcd}$$

This means that the subspace topology on S is the initial topology (on S) w.r.t. the inclusion.

Why?

- For 2 & 3 note that $i_S : S \xrightarrow{\cong} i_S(S) = S$ is simply the identity on S. This is continuous because $A \in \tau \Rightarrow i_S^{-1}(A) \in \tau_S$ since $i_S^{-1}(A) = A \cap S$.

- For 4, consider (Y, τ_Y) and $f : Y \to S$.

  a) $f : (Y, \tau_Y) \to (S, \tau_S)$ continuous implies $i_S \circ f : (Y, \tau_Y) \to (X, \tau)$ continuous. (Lemma 1.25)

  b) Now, assume that $i_S \circ f$ is continuous and note that

$$\mathcal{U} \in \tau_S \Rightarrow \mathcal{U} = V \cap S \quad \text{for some } V \in \tau$$
$$= i_S^{-1}(V) \quad \text{for some } V \in \tau$$

Thus, $f^{-1}(\mathcal{U}) = (i_S \circ f)^{-1}(V) \in \tau_Y$

and f is continuous. ☒

Rmk Uniqueness follows from the following:

Let $\tilde{\tau}_S$ be another topology on S satisfying this universal property. Then

$$\begin{tikzcd}
(S, \tilde{\tau}_S) \arrow[r, hook, "i_S"] & (X, \tau) \\
(Y, \tau_Y) = (S, \tau_S) \arrow[u, "f = id_S"] \arrow[ur] &
\end{tikzcd}$$

$i_S \circ f = i_S \circ id_S = i_S$

we know this is continuous

gives $\tilde{\tau}_S \subset \tau_S$ (since $i_S \circ f$ cont. $\Rightarrow$ f cont.).

That is, τ_S is finer (larger) than $\tilde{\tau}_S$.

Similarly,

$$\begin{tikzcd}
(S, \tilde{\tau}_S) \arrow[r, hook, "i_S"] & (X, \tau) \\
(S, \tilde{\tau}_S) \arrow[u, "f = id_S"] \arrow[ur] &
\end{tikzcd}$$

we know this is cont.

$i_S \circ f = i_S \circ id_S = i_S$

gives $i_S : (S, \tilde{\tau}_S) \hookrightarrow (X, \tau)$ is continuous.

(since f cont. $\Rightarrow i_S \circ f$ cont.)

But then this implies $\tau_S \subset \tilde{\tau}_S$ by minimality of τ_S (see 3).

Thus, τ_S is also coarser (smaller) than $\tilde{\tau}_S$. Hence, we have equality $\tau_S = \tilde{\tau}_S$.

Definition / proposition (product topology)

Let (X, τ_X) and (Y, τ_Y) be two top. spaces and consider

$$\mathcal{B} := \{ \mathcal{U} \times V \; ; \; \mathcal{U} \in \tau_X \text{ and } V \in \tau_Y \}.$$

Then $\mathcal{B}$ is a basis for a topology on $X \times Y$ and $\tau^{\mathcal{B}}$ (the topology generated by $\mathcal{B}$) is called the product topology on $X \times Y$.

Notation: $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$
$(x,y) \mapsto x$ $\qquad$ $(x,y) \mapsto y$

$\tau_{prod} := \tau^{\mathcal{B}}$

Properties Given (X, τ_X) and (Y, τ_Y),

① the product topology on $X \times Y$ is the **smallest** topology on $X \times Y$ for which <u>both</u> projections π_X and π_Y are continuous.

② The product topology on $X \times Y$ is the **unique** topology on $X \times Y$ for which the following universal property holds:

- If (Z, τ_Z) is a top. space, then $f : (Z, \tau_Z) \to (X \times Y, \tau_{prod})$ is continuous iff $\pi_X \circ f : (Z, \tau_Z) \to (X, \tau_X)$ and $\pi_Y \circ f : (Z, \tau_Z) \to (Y, \tau_Y)$

are continuous:

$$
\begin{tikzcd}
(Y,\tau_Y) & (X\times Y, \tau_{prod}) \arrow[l, "\pi_Y"'] \arrow[r, "\pi_X"] & (X,\tau_X) \\
& (Z,\tau_Z) \arrow[ul, "\pi_Y\circ f"] \arrow[u, "f"'] \arrow[ur, "\pi_X\circ f"'] &
\end{tikzcd}
$$

Exercise Let (X,τ_X) and (Y,τ_Y) be two top. spaces and let $f\colon (X,\tau_X)\to(Y,\tau_Y)$ be a continuous function.

Prove that $\Gamma := \mathrm{graph}(f) \subset X\times Y \xrightarrow{\pi_X|_\Gamma} X$

is a homeomorphism.

Here, on $X\times Y$ we consider the product topology and on $\Gamma = \mathrm{graph}(f)\subset X\times Y$ the subspace topology.

Recall $\Gamma = \text{graph}(f) = \{(x,y) \in X \times Y ;\ y = f(x)\}$

$$\text{and } \pi_X|_\Gamma : \Gamma \longrightarrow X$$
$$(x, f(x)) \longmapsto x$$

Now, let $\varphi = \pi_X|_\Gamma$. By property ① on p. 7, π_X is continuous. So, φ is continuous because is the restriction of a continuous function (① on p. 2).

Next, consider $\psi := \varphi^{-1} : X \longrightarrow \Gamma \subset X \times Y$.

Since both $\pi_X \circ \psi = id_X$ and $\pi_Y \circ \psi = f$ are continuous, property ② on p. 7 gives ψ is continuous as well.

$$\begin{tikzcd}
Y & \Gamma \subset X \times Y \arrow[l, "\pi_Y"'] \arrow[r, "\pi_X"] & X \\
& X \arrow[ul, "f"] \arrow[u, "\psi"'] \arrow[ur, "id_X"'] &
\end{tikzcd}$$

$\blacksquare$

Definition (disjoint union topology)

Given two top. spaces (X_1, τ_1) and (X_2, τ_2),

we define

$$X_1 \amalg X_2 := \{ (x,i) \, ; \, x \in X_i \}$$

This space comes with inclusion maps

$$\begin{aligned} f_i : X_i &\hookrightarrow X_1 \amalg X_2 \\ x &\longmapsto (x,i) \end{aligned}$$

and we define the disjoint union topology on $X_1 \amalg X_2$ by declaring

$$\mathcal{U} \subset X_1 \amalg X_2 \text{ open} \iff f_i^{-1}(\mathcal{U}) \text{ is open in } X_i \text{ for } i=1,2.$$

This is the largest topology on $X_1 \amalg X_2$ for which the f_i are continuous.

The quotient topology

Definition Given two sets X and Y, and a map $f: X \to Y$, we say $S \subset X$ is f-saturated if $S = f^{-1}(f(S))$.

Definition Let (X, τ_X) and (Y, τ_Y) be two topological spaces. A <u>surjective map</u> $\pi: X \to Y$ is called a QUOTIENT MAP iff

$$\mathcal{U} \in \tau_Y \iff \pi^{-1}(\mathcal{U}) \in \tau_X .$$

RmKs.:

1) $\pi: X \to Y$ continuous and surj. is a quotient map iff maps π-saturated opens of X to opens of Y.

2) If $\pi: X \to Y$ is a surjective continuous

map that is either an open or a closed map,

means $\mathcal{U} \in \tau_X \Downarrow \pi(\mathcal{U}) \in \tau_Y$

means $X \setminus \mathcal{U} \in \tau_X \Downarrow Y \setminus \pi(\mathcal{U}) \in \tau_Y$

then π is a quotient map.

⚠ $X = [0,1] \cup [2,3]$ as a subspace of $(\mathbb{R}, \tau_{usual})$

$Y = [0,2]$,,

$$\pi : X \longrightarrow Y$$
$$x \longmapsto \begin{cases} x & \text{if } x \in [0,1] \\ x-1 & \text{if } x \in [2,3) \end{cases}$$

This is surjective, continuous and closed, hence a quotient map. However, π is <u>not</u> an open map: $[0,1]$ is open on X and $\pi([0,1]) = [0,1]$ is <u>not</u> open on Y. Remember, we are considering the subspace topology...

Definition (quotient topology)

Let (X, τ_X) be a top. space, Y a set and $\pi: X \to Y$ a surjective map. We define a topology on Y by declaring $\mathcal{U} \subset Y$ open iff $\pi^{-1}(\mathcal{U})$ is open in X.

(i) this is the largest topology on Y making π continuous.

(ii) it is the unique topology on Y making π a quotient map.

This topology, τ_{quot}, is called the quotient topology determined by π.

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* The majority of examples come from the following.

- (X,τ) a top. space and Y = a partition of X into disjoint subspaces whose union is X.

  $\rightsquigarrow \exists$ equivalence relation $\sim$ on X s.t.

  $Y = X/\sim$ and we consider

$$\begin{aligned} \pi : X &\longrightarrow X/\sim \\ x &\longmapsto [x] \end{aligned}$$

- In fact, any quotient map $\pi : X \longrightarrow Y$ can be viewed as a map of the form $\pi : X \longrightarrow X/\sim$ by setting $x \sim x'$ on X $\Longleftrightarrow$ $\pi(x) = \pi(x')$. That is, the equivalence classes are the fibers of π.

* The universal property characterizing the quotient topology is the following.

Given $f : Y \to Z$,

f is continuous

$\Updownarrow$

$f \circ \pi$ is continuous

$$\begin{tikzcd} (Y, \tau_{quot}) \arrow[d, "f"'] & (X, \tau_X) \arrow[l, "\pi \text{ quot. map}"'] \arrow[dl, "f \circ \pi"] \\ (Z, \tau_Z) & \end{tikzcd}$$