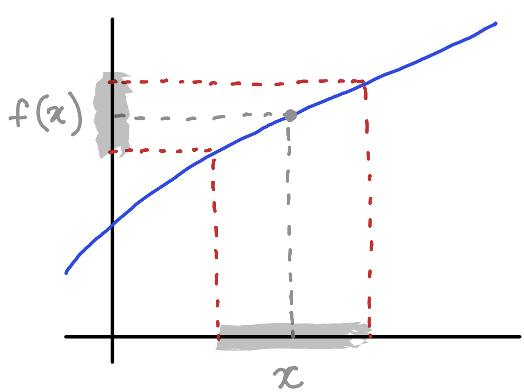
Math 220 - Lecture #3

Sep 24, 2024

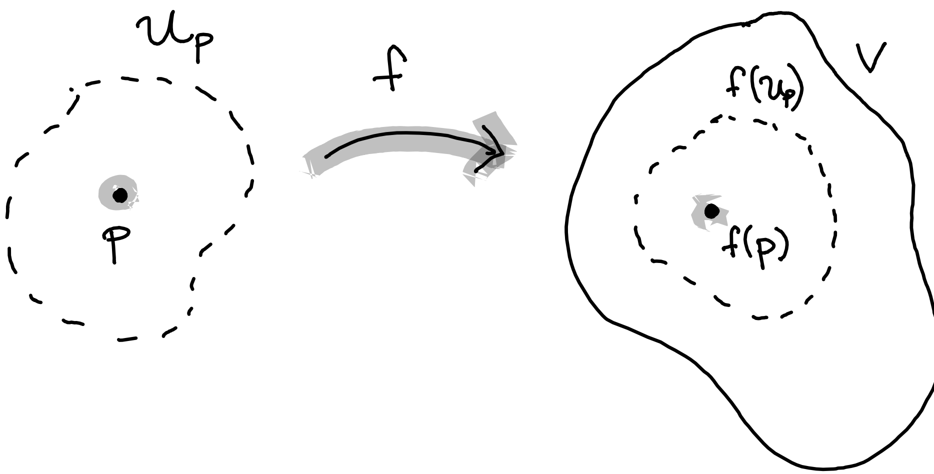
Today: $\begin{cases} \text{Convergence \& continuity} \\ \text{Homeomorphisms} \end{cases}$

Recall (from Analysis I) A function $f : \mathbb{R} \to \mathbb{R}$ is called continuous at a point $x \in \mathbb{R}$ (in the domain of f) iff given $\varepsilon > 0$, we can find $\delta > 0$ such that

$$|y - x| < \delta \implies |f(y) - f(x)| < \varepsilon$$



Definition Let (X, τ_x) and (Y, τ_Y) be two topological spaces. A function $f: X \to Y$ is called continuous at $p \in X$ iff given $V \in \tau_Y$ with $f(p) \in V$, we can find $\mathcal{U}_p \in \tau_X$ such that $p \in \mathcal{U}_p$ and $f(\mathcal{U}_p) \subset V$.



If f is continuous at all points $p \in X$, then we simply say f is continuous.

Proposition: A function $f:(X,\tau_X)\to(Y,\tau_Y)$ between two topological spaces is continuous if and only if

$$V\in\tau_Y \Rightarrow \mathcal{U}=f^{-1}(V)\in\tau_X \qquad \bigstar$$

("the pre-image of any open is open".)

proof.

($\Leftarrow$) Assume $\bigstar$ holds and pick $p\in X$.

If $V\in\tau_Y$ is such that $f(p)\in V$, then letting $\mathcal{U}_p:=f^{-1}(V)$ we have that $\mathcal{U}_p\in\tau_X$, $p\in\mathcal{U}_p$ and $f(\mathcal{U}_p)\subset V$.

Thus, f is continuous @ p.

($\Rightarrow$) Conversely, assume f is continuous and pick $V\in\tau_Y$. Let $\mathcal{U}:=f^{-1}(V)$.

We want to show $\mathcal{U}\in\tau_X$.

If $\mathcal{U} = \emptyset$ we are done. So, assume $\mathcal{U} \neq \emptyset$ and pick $p \in \mathcal{U}$. Since f is continuous @ p and $f(p) \in V$, we can find $\mathcal{U}_p \in \mathcal{T}_X$ such that $p \in \mathcal{U}_p$ and $f(\mathcal{U}_p) \subset V$.

Thus, $f^{-1}(V) = \mathcal{U} \supset f^{-1}(f(\mathcal{U}_p)) \supset \mathcal{U}_p$.

But then, since we can repeat the argument for all $p \in \mathcal{U}$, we deduce that

$$\mathcal{U} = \bigcup_{p \in \mathcal{U}} \mathcal{U}_p \quad \text{is open},$$

as it can be written as a union of opens. Hence, $\bigstar$ holds.

$\blacksquare$

Examples & non-examples

<u>Question</u> Are the functions below continuous?

(1) $\mathrm{id}: (X, \tau) \longrightarrow (X, \tau)$ ✓

(2) Fix $\blacktriangle \in X$ and consider

$$(X, \tau) \longrightarrow (X, \tau)$$
$$p \longmapsto \blacktriangle$$
✓

(3) Consider $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, \{a,b\}\}$ and $\tau' = \{\emptyset, \{b\}, \{a,b\}\}$. Look at

$$\mathrm{id}: (X, \tau) \longrightarrow (X, \tau')$$
✗

(4) $f: (X, \tau^{\text{disc}}) \longrightarrow (Y, \tau)$ ✓

(5) $g: (X, \tau) \longrightarrow (Y, \tau^{\text{indisc}})$ ✓

⑥ $(\mathbb{R}, \tau_{Euc}) \xrightarrow{\varphi} (\mathbb{Z}, \tau_{disc})$

$$p \longmapsto \lfloor p \rfloor$$

X

Definition Let (X, τ) be a top. space and (x_n) a sequence on X. We say (x_n) converges to $x \in X$, and we write $x_n \to x$, iff given $\mathcal{U} \in \tau$ with $x \in \mathcal{U}$, we can find $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow x_n \in \mathcal{U}$$

Proposition: If $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous and (x_n) is a seq. on X that converges to $x \in X$, then $(f(x_n))$ is a seq. on Y that converges to $f(x) \in Y$.

proof. Pick $V \in \tau_Y$ with $f(x) \in V$.

Since f is continuous @ x, $\exists\, \mathcal{U}_x \in \tau_X$ w/ $x \in \mathcal{U}_x$ & $f(\mathcal{U}_x) \subset V$.

Now, because $x_n \longrightarrow x$, for the open $\mathcal{U}_x$ we can find $N \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow x_n \in \mathcal{U}_x .$$

But then, $f(x_n) \in f(\mathcal{U}_x) \subset V$.

Thus,

$$n \geq N \Rightarrow f(x_n) \in V$$

and we conclude that $f(x_n) \longrightarrow f(x)$.

$\blacksquare$

Definition A function $f: (X, \tau_X) \to (Y, \tau_Y)$ is called sequentially continuous iff

$$x_n \to x \Rightarrow f(x_n) \longrightarrow f(x)$$

So, we have proved

$$f \text{ continuous} \implies f \text{ sequentially continuous}$$

⚠ The converse is not always true!

It is true for first countable spaces.
Among these we have the metric spaces (with their metric topology).

Definition Let (X, τ) be a top. space and fix $p \in X$. A collection $\mathcal{B}$ of elements of τ is called a local basis @ p iff

(i) $p \in B \quad \forall\, B \in \mathcal{B}$

(ii) if $\mathcal{U} \in \tau$ is such that $p \in \mathcal{U}$, then $\exists\, B \in \mathcal{B}$ with $p \in B \subset \mathcal{U}$

Example $(\mathbb{R}, \tau_{usual})$, any $p \in \mathbb{R}$.

$$\mathcal{B} := \{(a,b) \; ; \; a < p < b\}$$

Definition A top. space (X, τ) is called first countable if there exists a <u>countable</u> local basis at every point (of X).

Proposition Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be seq. cont. @ $x \in X$ and assume we can find a <u>countable</u> local basis @ x. Then f is continuous @ x.

Exercise Let (X, τ) be a first countable top. space and fix $A \subset X$. Prove that

$$a \in cl(A) \iff \exists \, (a_n) \text{ in } A \text{ s.t. } a_n \to a$$

When are two top. spaces "the same"?

Definition A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is called a HOMEOMORPHISM iff f is bijective and <u>both</u> f and f^{-1} are continuous.

Moreover, if that is the case we then say X and Y are homeomorphic (= "the same" as top. spaces).

Examples

① $X = \{0,1\}$ with $\tau = \{\phi, \{1\}, \{0,1\}\}$ is homeomorphic to X with $\tau' = \{\phi, \{0\}, \{0,1\}\}$ via

$$0 \longmapsto 1$$
$$1 \longmapsto 0$$

② $(0,1)$ and $\mathbb{R}$ with the usual topologies

In fact, next time we will learn that given $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ continuous, then $\text{graph}(f) := \{(x,y) \in X \times Y;\ y = f(x)\}$ is homeomorphic to X via 1st projection.

③

$$\{(x,y) \in \mathbb{R}^2;\ |x| + |y| = 1\} \sim_{\text{homeo}} \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 = 1\}$$

by considering

$$f : (x,y) \longmapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

A non-example

- $X = [0, 2\pi)$, τ_X = opens of $\mathbb{R}$ (usual top.) intersected with X
- $Y = S^1 = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2=1\}$

  τ_Y = opens of $\mathbb{R}^2$ (usual top.) intersected with Y

- $$\begin{aligned} f : X &\longrightarrow Y \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

Claim f is continuous and f is a bijection. However, $g = f^{-1}$ is not continuous @ $p = (1,0) \in Y$.

Idea

$$\begin{aligned} g : Y &\longrightarrow X \\ p = (1,0) &\longmapsto g(p) = 0 \end{aligned}$$

Take $V = X \cap (-\varepsilon, 1)$, $\quad 0 < \varepsilon \ll 1$

Then $V \in \tau_X$, $g(p) = 0 \in V$

but we <u>cannot</u> find $\mathcal{U}_p \in \tau_Y$, w/ $p \in \mathcal{U}_p$

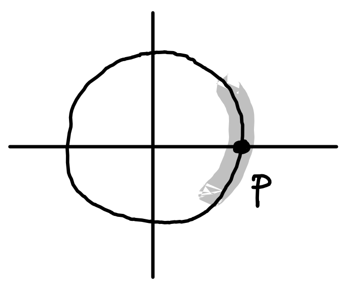
s.t. $g(\mathcal{U}_p) \subset V$.

this is what fails

Why?

Any $\mathcal{U}_p \in \tau_Y$ with $p \in \mathcal{U}_p$ must

contain an "arc" around p:



But any such arc contains a point

of the form $p_n := \left(\cos\left(2\pi - \frac{1}{n}\right), \sin\left(2\pi - \frac{1}{n}\right)\right)$

for some $n \in \mathbb{N}$ and

$$g(p_n) = 2\pi - \frac{1}{n} > 1$$