Math 220 - Lecture #2

Sep 17, 2024

Today { What is a topology?
Interior, closure & boundary

Definition Given a set X, a topology on X is a collection $\mathcal{T}$ of subsets of X satisfying the following:

T1 $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

Note that $X \setminus \emptyset \in \mathcal{T}$ and $X \setminus X \in \mathcal{T}$,

T2 Given $\{A_\alpha\}$ with each $A_\alpha \in \mathcal{T}$, we have that $\bigcup A_\alpha \in \mathcal{T}$

$$X \setminus \left(\bigcup A_\alpha\right) = \bigcap (X \setminus A_\alpha), \text{ and}$$

T3 $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$

Definition A topological space is a set X equipped with a topology τ. The elements of τ are called open.

Definition If (X, τ) is a topological space, then $F \subset X$ is called closed if $X \setminus F \in \tau$.

Question: Given (X,τ), can $A \subset X$ be open and closed at the same time? Can it be neither?

A list of examples

① DISCRETE TOPOLOGY

X any set and $\tau = \mathcal{P}(X)$ ← power set

② INDISCRETE TOPOLOGY

X any set and $\tau = \{\emptyset, X\}$

③ COFINITE TOPOLOGY

X any set and $\tau = \{ A \subset X ; \ X \setminus A \text{ is finite or } A = \emptyset \}$

(4) METRIC TOPOLOGY

Recall that a metric space (M, d) consists of a set M together with a function $d : M \times M \longrightarrow \mathbb{R}_{\geq 0}$ satisfying

M1 $\quad d(x,y) = 0 \iff x = y$

M2 $\quad d(x,y) = d(y,x)$

M3 $\quad d(x,z) \leq d(x,y) + d(y,z)$

for all $x, y, z \in M$. Such function d is called a metric on M.

Now, given a metric space (M, d), we can define a topology τ^d on M as follows. We declare $\mathcal{U} \subset M$ open iff

$$* \left\{ \begin{array}{l} \text{given } p \in \mathcal{U} \text{ we can find } \delta > 0 \text{ s.t.} \\ B(p, \delta) := \{ q \in M ; \ d(p,q) < \delta \} \subset \mathcal{U} . \end{array} \right.$$

That is, $\mathcal{U} \in \tau^d \iff \mathcal{U}$ satisfies $\star$

<u>Exercises</u>

1) Verify that all open balls are open.
2) Verify that all singletons are closed (T_1 property).

⑤ ZARISKI TOPOLOGY

Let $R = \mathbb{C}[x_1, \dots, x_n]$ and given $F \subset R$ define

$$D(F) := \{ x = (x_1, \dots, x_n) \in \mathbb{C}^n ; \exists f \in F \text{ s.t. } f(x) \neq 0 \}$$
$$= \mathbb{C}^n \setminus \{ x \in \mathbb{C}^n ; f(x) = 0 \ \forall f \in F \}$$

Then $\tau^{Zar} = \{ A \subset \mathbb{C}^n ; \exists F \subset R \text{ s.t. } A = D(F) \}$ is a topology on $\mathbb{C}^n$.

⑥ More generally, let R be a ring (commutative, $1 \in R$) and let $X = \mathrm{Spec}(R) = \{ I \leq R ;\ I \text{ is prime} \}$

We declare $F \subset X$ closed iff

$$F = V(J) := \{ I \in X ;\ J \subset I \}$$

for some $J \leq R$.

(An ideal $I \lneq R$ is called prime iff
$ab \in I \Rightarrow a \in I$ or $b \in I$)

proper

That is, we define

$$\tau = \{ A \subset X ;\ A = X \setminus V(J) \text{ for some } J \leq R \}$$

Definition Let X be a set. A collection $\mathcal{B}$ of subsets of X is called a basis for a topology on X iff

(i) given $x \in X$, $\exists\, B \in \mathcal{B}$ s.t. $x \in B$

(ii) if $x \in B_1 \cap B_2$, $B_i \in \mathcal{B}$, then
$\exists\, B \in \mathcal{B}$ s.t. $x \in B \subset B_1 \cap B_2$

Remark / definition Given X and $\mathcal{B}$ as above,

$$\tau^{\mathcal{B}} := \{ \mathcal{U} \subset X ;\ x \in \mathcal{U} \Rightarrow \exists\, B \in \mathcal{B} \text{ s.t. } x \in B \subset \mathcal{U} \}$$

is a topology. This is called the <u>topology generated by $\mathcal{B}$</u> and we say $\mathcal{B}$ is a basis for $\tau^{\mathcal{B}}$.

Examples

0) (X, τ) any topological space and
$\mathcal{B} = \{$ subsets of X s.t. any $A \in \tau$ is a union of sets in $\mathcal{B} \}$

Then $\tau^{\mathcal{B}} = \tau$

1) (M,d) a metric space with the metric topology τ^d.

Consider $\mathcal{B}$ = collection of all open balls.

Then $\tau^{\mathcal{B}} = \tau^d$.

In particular, every $\mathcal{U} \in \tau^d$ is a union of elements of $\mathcal{B}$ ($\iff$ every $\mathcal{U} \in \tau^d$ is a union of open balls.)

2) $(\mathbb{C}^n, \tau^{zar})$ and $\mathcal{B} = \{D(f) ; f \in R\}$

Interior, closure & boundary

In what follows, we fix (X, τ) a top. space.

Definition Given $A \subset X$, we say $x \in X$ is an interior point of A if we can find $\mathcal{U} \in \tau$ s.t. $x \in \mathcal{U} \subset A$.

Definition / proposition Given $A \subset X$,

$$\text{int}(A) = \{ x \in X \; ; \; x \text{ is an interior point of } A \}$$
$$= \text{union of all opens (of } X) \text{ contained in } A$$
$$= \text{largest open set contained in } A$$

Note: $\text{int}(A) \subset A$ and $A = \text{int}(A) \iff A \in \mathcal{T}$

Definition Given $A \subset X$, we say $x \in X$ is a **closure point** of A if given $\mathcal{U} \in \mathcal{T}$ with $x \in \mathcal{U}$ we have that $\mathcal{U} \cap A \neq \phi$.

Definition / proposition Given $A \subset X$,

$$cl(A) = \overline{A} = \{ x \in X \; ; \; x \text{ is a closure point of } A \}$$
$$= \text{intersection of all closed sets containing } A$$
$$= \text{smallest closed set that contains } A$$

Note: $A \subset cl(A)$ and $X \setminus A \in \mathcal{T} \Leftrightarrow A = cl(A)$

Definition Given $A \subset X$, the boundary of A is the set

$$\partial A := cl(A) \setminus int(A)$$

Proposition: Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.

$x \in \partial A \Longleftrightarrow$ given $\mathcal{U} \in \mathcal{T}$ with $x \in \mathcal{U}$ we have that $\mathcal{U} \cap A \neq \phi$ and $\mathcal{U} \cap (X \setminus A) \neq \phi$.

Or, equivalently,

$x \notin \partial A \Longleftrightarrow \exists\, \mathcal{U} \in \mathcal{T}$ with $x \in \mathcal{U}$ s.t. either $\mathcal{U} \cap A = \phi$ or $\mathcal{U} \cap (X \setminus A) = \phi$.

Proof

if $x \notin \partial A$, then either $x \in \text{int}(A)$ or $x \in X \setminus \text{cl}(A)$.

In the first case, take $\mathcal{U} = \text{int}(A)$.

Then $\mathcal{U} \in \tau$, $x \in \mathcal{U}$ and $\mathcal{U} \cap (X \setminus A) = \phi$ since $\mathcal{U} \subset A$.

In the second case, take $\mathcal{U} = X \setminus \text{cl}(A)$.

Again, $\mathcal{U} \in \tau$ and $x \in \mathcal{U}$. Moreover,

$\mathcal{U} \cap A = \phi$ since $\mathcal{U} \subset X \setminus A$.

Conversely, take $x \in X$ and assume $\exists \mathcal{U} \in \tau$ with $x \in \mathcal{U}$ s.t. either

(1) $\mathcal{U} \cap A = \phi$ or (2) $\mathcal{U} \cap (X \setminus A) = \phi$.

If (1) holds, then $X \setminus \mathcal{U}$ is a closed set that contains A. Thus,

$\text{cl}(A) \subset X \setminus \mathcal{U} \Rightarrow x \notin \text{cl}(A) \Rightarrow x \notin \partial A$

Finally, if ② holds, then

$$\mathcal{U} \subset A \Rightarrow \mathcal{U} \subset \mathrm{int}(A) \Rightarrow x \notin \underbrace{\mathrm{cl}(A) \setminus \mathrm{int}(A)}_{= \partial A}$$

That is, in both cases the conclusion is that $x \notin \partial A$.

$\boxtimes$