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MA A1 354

Math 220 - Topology I (point set topology)

Introductory lecture (Sep 10, 2024)

§0 - First things first

- Do you know how this course is organized?
- Do you know where to find information?
- If you have a question, you should ask! :)

§1 What is this course about?

SETS

* TOPOLOGICAL SPACES

Metric spaces

+ functions

The guiding questions will be the following:

1. When are two spaces considered the same?

2. What are the intrinsic properties of a topological space? What about metric spaces?

Our plan

Chapter 1 - Topological spaces and continuity

- Lectures 2, 3, 4 & 5

Chapter 2 - (simple) connectedness

- Lectures 6, 7 & 8

Chapter 3 - compactness

- Lectures 9 & 10

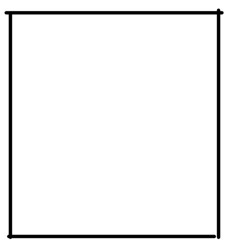
Chapter 4 - metric spaces

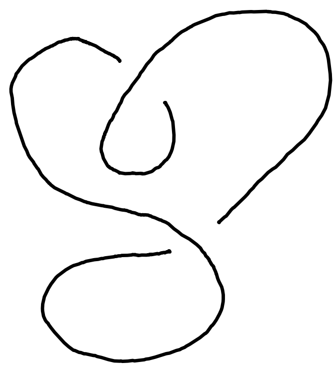
- Lectures 11, 12 & 13

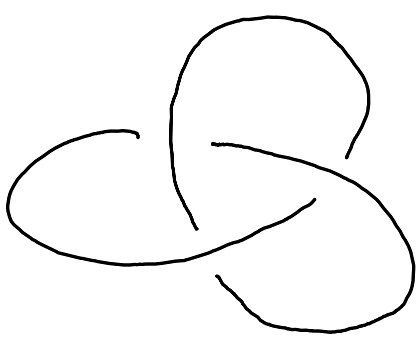
* Lecture # 14 on 17/12 will be a review session

§ 2 A warm-up

Question Are the following "spaces" different or equal, and why?







§3 Topics to be reviewed before next lecture

- You should read Chapter 0 in the lecture notes (by Juhan Aru)
- This course builds on your knowledge on sets, cartesian products, relations, functions (injection, surjection, bijection), cardinality...
- You should recollect the statements (and proofs) of the following theorems:

  Bolzano-Weierstrass,
  Heine-Borel,
  Extreme value theorem, and
  Intermediate value theorem.

We will revisit these theorems within the world of topological spaces.

For completeness, the statements are presented below.

B-W (for $\mathbb{R}$) Every bounded sequence of real numbers admits a convergent subsequence.

H-B (for $\mathbb{R}$) Given $A \subset \mathbb{R}$, T.F.A.E :

(i) A is compact (every open cover has a finite subcover)

(ii) A is closed and bounded

Extreme Value Theorem

$f : [a,b] \to \mathbb{R}$ continuous $\Rightarrow \exists\, c, d \in [a,b]$ such that $f(c) \le f(x) \le f(d) \quad \forall\, x \in [a,b]$.

Intermediate Value Theorem

$f: [a,b] \longrightarrow \mathbb{R}$ continuous, $f(a) \leq y \leq f(b)$

$\Rightarrow \exists\ c \in [a,b]$ such that $f(c) = y$.

(here, w.l.o.g. we assume $f(a) \leq f(b)$)

§4 - An application of the Cantor - Schroeder - Bernstein theorem

Recall (CSB) Let A and B be two arbitrary sets. If there exist injections $f: A \to B$ and $g: B \to A$, then a <u>bijection</u> $h: A \to B$ exists.

Question Are $\mathbb{R}^2$ and $\mathbb{R}$ the same?

Of course the answer depends on what do we mean by "the same" and on which "category" we are considering...

But, today we will see $\mathbb{R}^2 \sim \mathbb{R}$ as sets (meaning that they have the same cardinality). Later in the course we will learn that $\mathbb{R}^2 \not\sim \mathbb{R}$ as topological spaces (w.r.t. the usual Euclidean topology).

The idea

1. $A \sim B \iff \exists$ bijection $f: A \to B$
2. $A \sim C$ and $B \sim D$, then $A \times B \sim C \times D$
3. $\mathbb{R} \sim (0,1)$
4. $(0,1) \times (0,1) \sim (0,1)$ $\Leftarrow$

To prove ④, by CSB, it suffices to construct injections $f: (0,1) \to (0,1) \times (0,1)$ and $g: (0,1) \times (0,1) \to (0,1)$.

For example, take $f: x \mapsto (x, \bullet)$, where $\bullet$ is any fixed element of $(0,1)$.

What about g ?

Trick Represent an element $x \in (0,1)$ by its decimal expansion $0.x_1 x_2 x_3 \ldots$ and do not allow infinite strings of 9s

So, $1/2 = 0.5$ and <u>not</u> $0.49999\ldots$

In other words, $x = \sum_{i=1}^{\infty} x_i / 10^i$

and for each k, x_k is the <u>largest</u> integer such that

$$\frac{x_1}{10} + \cdots + \frac{x_k}{10^k} \leq x$$

Then, given $x = 0.x_1 x_2 x_3 \cdots$ and $y = 0.y_1 y_2 y_3 \cdots$

we can let $g\big((x,y)\big) = 0.x_1 y_1 x_2 y_2 x_3 y_3 \cdots$

The function $g : (0,1) \times (0,1) \longrightarrow (0,1)$

defined this way is injective.

Remark to prove ③ you can, for instance,

consider the function $\varphi : \mathbb{R} \longrightarrow (0,1)$

$$x \longmapsto \frac{1}{1+e^x}$$