A general comment: We have been using the following facts quite often.

(A) (Exercise in Lec #3) Let (X, τ) be a first countable space and fix $A \subset X$.

Then $a \in cl(A) \iff \exists\, (a_n)$ in A s.t. $a_n \longrightarrow a$.

<u>Corollary</u>: $A \subset X$ is closed $\iff$ given (a_n) in A with $a_n \to a$, we have that $a \in A$.

In general, $a_n \to a \Rightarrow a \in cl(A)$
$\Leftarrow$ true if 1st countable

(B) Metric spaces are first countable.

Math 220 - Lecture #13

Dec 17, 2024

Today
- "Topological sizes"
- Baire's category theorem
- A criterion for metrizability (Urysohn)

Definition Let (X, τ) be a topological space. We say that $A \subset X$ is <u>nowhere dense</u> if $\text{int}(\text{cl}(A)) = \emptyset$.

Examples

0. The empty set. If $\tau = \tau_{dis}$ this is the only example.
1. $\{1/n;\ n \in \mathbb{N}\} \subset \mathbb{R}$ OR $\mathbb{Z} \subset \mathbb{R}$
2. The Cantor set as a subset of $\mathbb{R}$. (see exercise 1.26)
3. Any countable union of lines in $\mathbb{R}^2$.
4. On a <u>metric space</u> with no isolated points, any finite subset is nowhere dense.

* a point $x \in X$ is isolated $\Leftrightarrow \{x\} \in \tau$

Definition Let (X, τ) be a topological space. A subset $A \subset X$ is called <u>meagre</u> if it can be written as a countable union of nowhere dense subsets of X. If $A \subset X$ is meagre, then $X \setminus A$ is called residual or generic.

Definition A topological space (X, τ) is called a <u>Baire space</u> if any meagre subset $A \subset X$ has empty interior.

Example: Any compact Hausdorff top. space.

|| One of our goals today is to prove the following result.

Theorem (Baire category theorem) Every <u>complete</u> metric space (M, d) is a Baire space (w/ the metric topology).

Two important consequences

① If (M, d) is complete $(M \neq \phi)$, then M cannot be written as a countable union of nowhere dense subsets.

② If (M,d) is complete $(M \neq \emptyset)$ and M can be written as a countable union of closed subsets, then at least one of these closed subsets must contain an open ball.

In the proof of the theorem we will use the following two lemmas.

Lemma 1 (Cantor) A metric space (M,d) is complete iff for any nested chain of closed subsets $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$ such that $\lim_{n\to\infty} \operatorname{diam}(C_n) = 0$ there exists $p \in M$ with $\bigcap_n C_n = \{p\}$.

Here, for $A \subset M$ we define

$$\operatorname{diam}(A) = \sup\{d(x,y);\ x,y \in A\}$$

Lemma 2 A topological space (X, τ) is a Baire space iff any countable intersection of dense open subsets is dense (in X).

proof of the theorem

Pick $\{\mathcal{U}_i\}$ countable with each $\mathcal{U}_i \in \tau^d$ and dense. By Lemma 2, it suffices to prove that $\mathcal{U} := \bigcap \mathcal{U}_i$ is dense.

Fix any open ball $B_1 \subset M$. Since $\mathcal{U}_1$ is open and dense, $B_1 \cap \mathcal{U}_1$ is both open and non-empty. Thus, by the definition of τ^d, $\exists$ an open ball $B_2 \subset M$ s.t. $\overline{B_2} \subset B_1 \cap \mathcal{U}_1$

&

The corresp. closed ball

w.l.o.g we may assume that B_2 has radius $< 1/2$.

Now, since $\mathcal{U}_2$ is open and dense, $B_2 \cap \mathcal{U}_2$ is both open and non-empty. Thus, $\exists$ an open ball $B_3 \subset M$ of radius $< 1/3$ s.t. $\overline{B_3} \subset B_2 \cap \mathcal{U}_2$.

Proceeding inductively, we obtain this way a chain $\overline{B}_1 \supset \overline{B}_2 \supset \cdots \overline{B}_n \supset \cdots$

of nested closed sets with $\overline{B_{i+1}} \subset B_i \cap \mathcal{U}_i$ and such that $\operatorname{diam} \overline{B}_n \to 0$ as $n \to \infty$.

By Lemma 1, since (M,d) is complete, $\exists\, p \in M$ with $\bigcap_n \overline{B}_n = \{p\}$.

But then $p \in \bigcap \mathcal{U}_i = \mathcal{U}$ and $p \in B_1$.

That is, $\mathcal{U} \cap B_1 \neq \emptyset$ and it follows that $\mathcal{U}$ is dense.

∎

A criterion for metrizability

<u>Theorem</u> (Urysohn) Let (X,τ) be a topological space. If (X,τ) is Hausdorff, normal and it has a countable basis, then (X,τ) is metrizable.

That is, $\exists$ a metric d on X such that $\tau^d = \tau$.

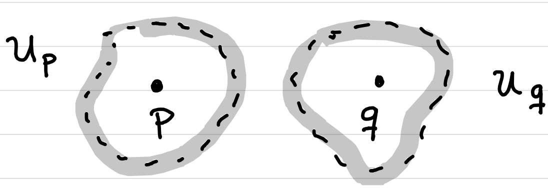
Non-examples

- If X is any set with more than 2 points, then (X, τ_{ind}) is not metrizable.

- If (X, τ) is not first-countable, then it is not metrizable. For example, $X = \mathbb{R}$ and $\tau =$ co-finite topology.

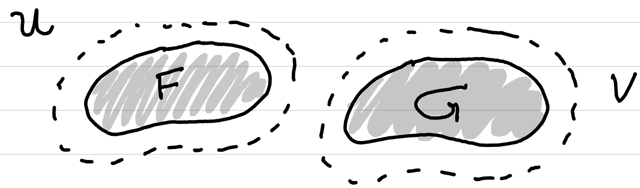
Definition (Hausdorff)

(X, τ) is Hausdorff iff given any two (distinct) points $p, q \in X$, we can find $\mathcal{U}_p, \mathcal{U}_q \in \tau$ s.t. $\mathcal{U}_p \cap \mathcal{U}_q = \phi$, $p \in \mathcal{U}_p$ and $q \in \mathcal{U}_q$.



Definition (Normal)

(X, τ) is normal iff given $F, G \subset X$ closed and disjoint, we can find $\mathcal{U}, V \in \tau$ s.t. $\mathcal{U} \cap V = \phi$, $F \subset \mathcal{U}$ and $G \subset V$.



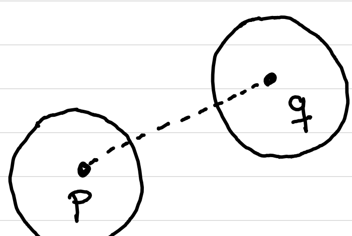
Proposition Any metric space (M, d) is both Hausdorff and normal.

Proof

(H) Given $p, q \in M$, $p \neq q$, let $\varepsilon = \frac{d(p,q)}{3}$

and let $\mathcal{U}_p = B(p, \varepsilon)$ and $\mathcal{U}_q = B(q, \varepsilon)$.

Then $\mathcal{U}_p \cap \mathcal{U}_q = \phi$, $p \in \mathcal{U}_p$ and $q \in \mathcal{U}_q$.



(N) Fix $F, G \subset M$ closed & disjoint.

For each $x \in F$, let $\varepsilon_x = d(x, G) > 0$;

and for each $y \in G$, let $\delta_y = d(y, F) > 0$.

Consider $\mathcal{U} = \bigcup_{x \in F} B(x, \varepsilon_x/3)$ and $V = \bigcup_{y \in G} B(y, \delta_y/3)$

Then $\mathcal{U}, V \in \tau^d$, $F \subset \mathcal{U}$ and $G \subset V$.

We want to show that $\mathcal{U} \cap V = \phi$.

By contradiction, assume that $\exists\, p \in \mathcal{U} \cap V$.

Then $\exists\, x \in F$ and $\exists\, y \in G$ such that

$p \in B(x, \varepsilon_x/3) \cap B(y, \delta_y/3)$. But then

letting $M = \max\{\varepsilon_x, \delta_y\}$, we have that

$$M \leq d(x,y) \leq d(x,p) + d(p,y)$$
$$< \frac{\varepsilon_x}{3} + \frac{\delta_y}{3} \leq \frac{2M}{3}$$

which is an absurd.

$\blacksquare$

Idea of the proof of Urysohn's thm:

(non-examinable)

- Since (X, τ) is normal, given $F, G \subset X$ closed and disjoint, $\exists\ f : (X, \tau) \to ([0,1], \tau_{usual})$ continuous such that $f(F) = \{0\}$ and $f(G) = \{1\}$ (This is known as Urysohn's lemma)
- Since (X, τ) is Hausdorff and has a countable basis, we can find $\{(\mathcal{U}_i, V_i)\}_{i \in \mathbb{N}}$ s.t. the $\mathcal{U}_i$ and the V_i generate τ and $cl(\mathcal{U}_i) \subset V_i \quad \forall i$
- Each pair $(\mathcal{U}_i, V_i)$ defines two disjoint closed sets $F_i = cl(\mathcal{U}_i)$ and $G_i = X \setminus V_i$, hence a continuous function $f_i : X \to [0,1]$ and one can consider

$$f : X \longrightarrow [0,1]^{\mathbb{N}}$$
$$x \longmapsto (f_1(x), f_2(x), \cdots)$$

- Now, $[0,1]^{\mathbb{N}}$ is naturally a metric space. and it suffices to show that f is a homeomorphism. (metrizability is a topological invariant)