Math 220 - Lecture # 12

Dec 10, 2024

Today: • Continuous functions with values in metrics spaces
• Arzelà-Ascoli

Notation:

- (X, τ) a topological space
- (M, d) a metric space
- $C(X, M) := \{ f : X \to M ;\ f \text{ is continuous} \}$
- $F(X, M)$ = space of all functions $f : X \to M$

FACT 1 We can endow $C(X, M)$ with the uniform/sup metric:

$$d_\infty(f, g) := \sup_{x \in X} d(f(x), g(x))$$

* If X is <u>compact</u>, then this is indeed a metric.

* In general, one considers $C(X, M) \subset F(X, M)$ and the so-called "truncated sup metric":

$$\bar{d}_\infty(f, g) := \min\{ d_\infty(f, g), 1 \}$$

Today, we will prove the following.

THEOREM (Thm 4.25 + Lemma 4.26 + Prop 4.27)

(i) $C(X,M)$ is closed in $(F(X,M), \bar{d}_\infty)$

(ii) If (M,d) is complete, then $(F(X,M), \bar{d}_\infty)$ is complete. In particular, by (i), $(C(X,M), \bar{d}_\infty)$ is complete.

(iii) If (X,τ) is compact and (M,d) is complete, then $(C(X,M), d_\infty)$ is complete.

Remarks

- In the theorem above, (i) can be rephrased as follows.

" If $(f_n) \in C(X,M)$ and $\bar{d}_\infty(f_n, f) \to 0$ for some $f \in F(X,M)$, then $f \in C(X,M)$."

- In a complete metric space, closed subsets are also complete.

- If (X,τ) is compact, then $\bar{d}_\infty$ and d_∞ are equivalent metrics on $C(X,M)$. But they are not necessarily Lipschitz equivalent.

- Recall that

Definition A sequence of functions $f_n : X \to M$ converges uniformly to a function $f: X \to M$ iff given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow d(f_n(x), f(x)) < \varepsilon \quad \forall x \in X$$

Thus, if $(f_n) \in C(X,M)$ and (X, τ) is compact,

$$f_n \xrightarrow{u} f \iff d_\infty(f_n, f) \longrightarrow 0$$

converges uniformly

In other words, uniform convergence is the same as convergence with respect to the uniform metric.

proof of theorem

(i) ("the $\varepsilon/3$ trick") Take $(f_n) \in C(X,M)$ with $f_n \to f$ in $(F(X,M), d_\infty)$. We want to show that $f \in C(X,M)$, i.e., that f is continuous.

Fix $x_0 \in X$. Since $f_n \longrightarrow f$ in the $\overline{d}_\infty$ metric, given $0<\varepsilon<1$, $\exists\, N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \underbrace{\overline{d}_\infty(f_n, f)}_{\displaystyle \min\left\{ \sup_{x\in X} d(f_n(x), f(x)),\, 1 \right\}} < \varepsilon/3$$

Thus, $\exists\, N \in \mathbb{N}$ large enough such that

$$n \geq N \Rightarrow d\big(f_n(x), f(x)\big) < \varepsilon/3 \quad \forall\, x \in X \qquad (1)$$

Now, each f_n is continuous @ x_0. So, $\exists\, \mathcal{U} \in \mathcal{T}$ with $x_0 \in \mathcal{U}$ s.t. $f_n(\mathcal{U}) \subset B\big(f_n(x_0), \varepsilon/3\big)$

$$\Updownarrow$$

$$(2) \quad x \in \mathcal{U} \Rightarrow d\big(f_n(x), f_n(x_0)\big) < \varepsilon/3$$

We claim that $f(\mathcal{U}) \subset B\big(f(x_0), \varepsilon\big)$, hence f is continuous @ x_0.

In fact, if $x \in \mathcal{U}$, then for $n \geq N$:

$$d\big(f(x), f(x_0)\big) \leq \underbrace{d\big(f(x), f_n(x)\big)}_{<\,\varepsilon/3 \text{ by } (1)} + \underbrace{d\big(f_n(x), f_n(x_0)\big)}_{<\,\varepsilon/3 \text{ by } (2)} + \underbrace{d\big(f_n(x_0), f(x_0)\big)}_{<\,\varepsilon/3 \text{ by } (1)}$$

$$< \varepsilon$$

That is, if $x \in \mathcal{U}$, then

$$d(f(x), f(x_0)) < \varepsilon \iff f(x) \in B(f(x_0), \varepsilon)$$

$$\iff f(\mathcal{U}) \subset B(f(x_0), \varepsilon).$$

$\blacksquare$

(ii) Pick $(f_n) \in F(X, M)$ a Cauchy sequence.

We want to show that $\exists f \in F(X,M)$ s.t.

$f_n \to f$ w.r.t. the $\overline{d}_\infty$ metric.

Now, given $x \in X$, $(f_n(x))$ is a Cauchy WHY?

sequence in (M, d). Since (M, d) is complete,

$\exists\, y_x \in M$ s.t. $f_n(x) \to y_x$. This defines

a function
$$X \xrightarrow{f} M$$
$$x \longmapsto f(x) := y_x = \lim_{n\to\infty} f_n(x)$$

We claim that $f_n \to f$. (Exercise)

$\blacksquare$

* In fact, one can show that (f_n) is

uniformly Cauchy. That is, given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$$n, m \geq N \implies d(f_n(x), f_m(x)) < \varepsilon \quad \forall x \in X$$

So, taking the limit as $m \to \infty$, we obtain $f_n \to f$

(in the $\overline{d}_\infty$ metric)

Lemma 1 Assume that (X, τ) is compact. Given $f \in C(X, M)$ and $\varepsilon > 0$, $\exists\, \mathcal{U}_1, \dots, \mathcal{U}_n \in \tau$ such that $X = \bigcup \mathcal{U}_i$ and

$$* \qquad x, y \in \mathcal{U}_i \Rightarrow d(f(x), f(y)) < \varepsilon$$

* This means that f is uniformly continuous.

<u>proof</u> Given any $p \in f(X) \subset M$, let $\mathcal{U}_p := f^{-1}\big(B(p, \varepsilon/2)\big)$. Since f is continuous, each $\mathcal{U}_p$ is open. Now, $X = \bigcup_{p \in f(X)} \mathcal{U}_p$ and

$\big(x \in X \Rightarrow x = f^{-1}(p)$ for some $p \in f(X)\big)$

since (X, τ) is compact, $\exists\, p_1, \dots, p_n \in f(X)$ such that $X = \bigcup \mathcal{U}_{p_i}$. Letting $\mathcal{U}_i = \mathcal{U}_{p_i}$, this cover will satisfy $*$, by construction.

Definition Fix $\mathcal{F} \subset C(X, M)$

(i) $\mathcal{F}$ is equicontinuous iff given $\varepsilon > 0$ we can find $\mathcal{U}_1, \dots, \mathcal{U}_n \in \tau$ s.t. $X = \bigcup \mathcal{U}_i$ and

$$x, y \in \mathcal{U}_i \Rightarrow d(f(x), f(y)) < \varepsilon \quad \forall f \in \mathcal{F}$$

(ii) $\mathcal{F}$ is uniformly bounded iff we can find an open ball $B \subset M$ such that $\forall x \in X$ and $\forall f \in \mathcal{F}$ we have that $f(x) \in B$.

Proposition (Prop 4.33)

If (X, τ) is compact and (f_n) is a sequence in $C(X, M)$ such that $f_n \xrightarrow{u} f$ for some $f \in C(X, M)$, then $\mathcal{F} = \{f\} \cup \{f_n ; n \in \mathbb{N}\}$ is equicontinuous and uniformly bounded.

of equicontinuity

proof Since f is continuous and X is compact, by Lemma 1, given $\varepsilon > 0$, $\exists\, \mathcal{U}_1, \dots, \mathcal{U}_n \in \tau$ s.t. $X = \bigcup \mathcal{U}_i$ and $x, y \in \mathcal{U}_i \Rightarrow d(f(x), f(y)) < \varepsilon/3$.

Fix $\varepsilon > 0$ and $\{\mathcal{U}_i\}$ (depending on $\varepsilon > 0$) as above.

Since $f_n \xrightarrow{u} f$, for this $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ (which depends on $\varepsilon > 0$) s.t.

$$n \geq N \Rightarrow d_\infty(f_n, f) < \varepsilon/3$$

Thus, $\forall\, x, y \in \mathcal{U}_i$ we have that

$n \geq N \Rightarrow d\big(f_n(x), f_n(y)\big) \leq d\big(f_n(x), f(x)\big) +$

$d\big(f(x), f(y)\big) + d\big(f(y), f_n(y)\big) < \varepsilon$

$\big($ Note : $\forall\, p \in X$, $d(f_n(p), f(p)) \leq d_\infty(f_n, f)$ $\big)$

This means that for this $\varepsilon > 0$, the cover $\{U_i\}$ works for $\mathcal{F}_0 = \{f\} \cup \{f_n ;\, n \geq N\}$

Now, we want a cover of X that works for $\mathcal{F} = \mathcal{F}_0 \cup \{f_1, \ldots, f_{N-1}\}$. Roughly, the idea is:

Since, for $j = 1, \ldots, N-1$, each $\{f_j\}$ is equicontinuous (Lemma 1) so is their finite union (Lemma 4.32 in the notes) hence so is $\mathcal{F}$. One has to be careful since N is depending on ε!

Uniform boundedness is left as an exercise. ∎

Here is how to construct this cover. For the $\varepsilon > 0$ we fixed, $\exists\, V_1^{(j)}, \ldots, V_{n(j)}^{(j)} \in \tau$ s.t.

$X = \bigcup V_{k(j)}^{(j)}$ and

We apply Lemma 1 to each j

$x, y \in V_{k(j)}^{(j)} \Rightarrow d\big(f_j(x), f_j(y)\big) < \varepsilon$

Now, consider $W_I = \mathcal{U}_i \cap V_{i_1}^{(1)} \cap V_{i_2}^{(2)} \cap \cdots \cap V_{i_{N-1}}^{(N-1)}$ for each $I = (i, i_1, \ldots, i_{N-1})$ where $i \in \{1, \ldots, n\}$ and $i_j \in \{1, \ldots, n(j)\}$. Then

$$X = \bigcup_I W_I \quad \& \quad x, y \in W_I \Rightarrow d(g(x), g(y)) < \varepsilon \quad \forall g \in \mathcal{F}.$$

* Here is the idea for uniform boundedness:

(X, τ) compact & f continuous $\Rightarrow$ f bounded $\Rightarrow \exists\, c > 0$ & $p \in M$ s.t. $d(f(x), p) < c$ $\forall x \in X$ $\Leftrightarrow$ $f(X) \subset B(p, c)$.

Now, for any $\delta > 0$, $\exists\, N_\delta \in \mathbb{N}$ such that $n \geq N_\delta \Rightarrow d(f_n(x), f(x)) < \delta \quad \forall x \in X$.

Thus,

$$n \geq N_\delta \Rightarrow d(f_n(x), p) \leq d(f_n(x), f(x)) + d(f(x), p) < c + \delta \quad \forall x \in X$$

$\Rightarrow$ We may assume that $\mathcal{F}_0$ is unif. bounded. & applying Lemma 4.32, we deduce that $\mathcal{F}$ is unif. bounded.

Theorem (Arzelà-Ascoli)
(thm 4.35)

Let (X, τ) be a compact topological space and let (M, d) be a metric space such that all closed balls are compact. Then $\mathcal{F} \subset C(X, M)$ is uniformly bounded and equicontinuous iff $\mathrm{cl}(\mathcal{F}) \subset C(X, M)$ is compact.

* A more "standard" statement is the following. Under the same hypotheses, if (f_n) is a seq. in $(C(X,M), d_\infty)$ which is uniformly bounded and equicontinuous, then it admits a convergent subsequence.

Rmk In the proof, we will use that the underlined assumption implies that (M, d) is complete, hence $C(X, M)$ is complete, too.

Sketch of the proof of $(\Rightarrow)$

Take $\mathcal{F} \subset C(X,M)$ uniformly bounded and equicontinuous. We want to show that $cl(\mathcal{F})$ is compact.

From Lecture #11, it suffices to show that $cl(\mathcal{F})$ is complete and totally bounded.

Now,

① (M,d) complete $\Rightarrow$ $C(X,M)$ complete (page 2) and since $cl(\mathcal{F})$ is closed, $cl(\mathcal{F})$ is complete. (rmk page 2)

② $\mathcal{F}$ is totally bounded $\Leftrightarrow$ $cl(\mathcal{F})$ is totally bounded

So, we will show that $\mathcal{F}$ is totally bounded.

Pick $\varepsilon > 0$. By equicontinuity, $\exists\ \mathcal{U}_1, \dots, \mathcal{U}_n \in \mathcal{T}$ such that $X = \bigcup \mathcal{U}_i$ and

$$x, y \in \mathcal{U}_i \Rightarrow d(f(x), f(y)) < \varepsilon/3 \qquad \forall f \in \mathcal{F}$$

By uniform boundedness, $\exists\, c>0$ & $p \in M$ s.t.

$\forall f \in \mathcal{F}$ we have that $f(X) \subset B(p,c)$

Since $\overline{B(p,c)}$ is compact (by assumption), hence

totally bounded, $\exists\, p_1, \dots, p_K \in M$ s.t.

$$B(p,c) \subset \overline{B(p,c)} \subset B(p_1, \varepsilon/6) \cup \cdots \cup B(p_K, \varepsilon/6)$$

Choose $x_j \in \mathcal{U}_j$ for $j = 1, \dots, n$ and $f_0 \in \mathcal{F}$,

and for each $I = (i_1, \dots, i_n) \subset \}1, \dots, K\{^n$

define

$$\begin{cases} f_I = f & \text{if } \exists f \in \mathcal{F} \text{ s.t. } f(x_j) \in B(p_{i_j}, \varepsilon/6)\ \forall j \\ f_I = f_0 & \text{otherwise} \end{cases}$$

Then, by construction, $\mathcal{F} = \bigcup_I B(f_I, \varepsilon)$

ball in $C(X, M)$

$\blacksquare$