Math 220 - Lecture # 11

Dec 3rd, 2024

Today: Compactness vs. completeness

Throughout these notes, let (M, d) be a metric space.

Definition We say that (M, d) is <u>sequentially compact</u> if every sequence of points of M admits a convergent subsequence.

One of our goals today is to prove the following result.

Theorem (= Prop. 4.11 + 4.15)

(M, d) is compact $\Leftrightarrow$ (M, d) is sequentially compact

w.r.t. τ^d

For the proof we will need three auxiliary lemmas.

Lemma 1 (Lemma 4.12) If (M, τ^d) is compact, then given any $\varepsilon > 0$ we can find $p_1, \dots, p_K \in M$ such that $M = \bigcup_{i=1}^{K} B(p_i, \varepsilon)$.

Why?

Lemma 2 (Lemma 4.14) If (M, d) is sequentially compact, then given an open cover $M = \bigcup \mathcal{U}_i$ we can find $\varepsilon > 0$ such that for each $p \in M$, $B(p, \varepsilon) \subset \mathcal{U}_i$ for some i.

Lemma 3 (Claim 4.16) If (x_n) is a sequence in M such that for some $\varepsilon > 0$ $d(x_i, x_j) > \varepsilon$ whenever $i \neq j$, then (x_n) admits no convergent subsequence.

proof of theorem

($\Rightarrow$) Assume that (M, τ^d) is compact and pick a sequence (x_n) in M. We want to show that (x_n) admits a convergent subsequence.

Since (M, τ^d) is compact, by Lemma 1, we can write $M = \bigcup_{i=1}^{K} B(p_i, 1/2)$ for some $p_1, \dots, p_K \in M$. Moreover, up to relabelling, we may assume that there exists infinitely many $n \in \mathbb{N}$ such that $x_n \in B_1 := B(p_1, 1/2)$.

Let n_1 be the smallest $n \in \mathbb{N}$ with this property. $\left(n_1 := \min \{ n \in \mathbb{N};\ x_n \in B(p_1, 1/2) \} \right)$

Now, applying Lemma 1 again with $\varepsilon = 1/4$ we can write $B_1 \subset \bigcup_{j=1}^{K'} B(q_j, 1/4)$ for some $q_1, \dots, q_{K'} \in B_1$. And, again, we may assume that $B_2 := B_1 \cap B(q_1, 1/4)$ contains infinitely many elements of (X_n). So, let $n_2 > n_1$ be the smallest $n \in \mathbb{N}$ such that $x_n \in B_2$.

Proceeding this way, we obtain a sequence of nested closed sets $C_i := \mathrm{cl}(B_i) =$

$$= \mathrm{cl}\left(\bigcap_{j=1}^{i} B(\tilde{p}_j, 1/2^j)\right),$$

where $\tilde{p}_1 = p_1$ and $\tilde{p}_2 = q_1$. Furthermore, we obtain a subsequence (x_{n_i}) of (x_n) such that $x_{n_i} \in C_i$.

But then, by Cantor's lemma (Lecture #8), $\bigcap_i C_i \neq \phi$ and, by construction, if $x \in \bigcap C_i$, then $x_{n_i} \longrightarrow x$.

($\Leftarrow$) Assume now that (M, d) is sequentially compact and pick an open cover $M = \bigcup \mathcal{U}_i$ $(\mathcal{U}_i \in \tau^d)$. We want to show that there exist finitely many of these opens $\mathcal{U}_i$ that cover M.

By Lemma 2, $\exists\, \varepsilon > 0$ s.t. for each $p \in M$, $B(p, \varepsilon) \subset \mathcal{U}_i$ for some $i = i(p)$

Thus, if we can show that for this $\varepsilon > 0$,

$$M = \bigcup_{j=1}^{K} B(p_j, \varepsilon), \text{ for some } p_1, \dots, p_K \in M,$$

then it will follow that the sets $\mathcal{U}_j = \mathcal{U}_{i(p_j)}$ form a finite subcover.

Assume, by contradiction, that is not the case. Then picking any $p_1 \in M$ and, inductively,

$$p_n \in M \setminus \bigcup_{i=1}^{n-1} B(p_i, \varepsilon) \quad \text{for all } n \in \mathbb{N},$$

we construct a sequence $(p_n) \in M$ as in Lemma 3, which contradicts our assumption that M is sequentially compact.

$\blacksquare$

Completeness

Recall Let (x_n) be a sequence in (M,d).

1. (x_n) is Cauchy iff given $\varepsilon>0$, $\exists\, N\in\mathbb{N}$ s.t.

$$n,m\geq N \Rightarrow d(x_n,x_m)<\varepsilon .$$

(Def 4.17)

2. If (x_n) is Cauchy and some subsequence (x_{n_k}) converges to some $x\in M$, then $x_n \longrightarrow x$.

   (This is claim 4.21 in the notes)

3. If (x_n) converges, then (x_n) is Cauchy.

   (This Lemma 4.18)

4. If (x_n) is such that $\exists$ two subsequences converging to two different points of M, then the sequence cannot be Cauchy.

Definition We say that (M,d) is complete if any Cauchy sequence in M converges (to a point of M).

Examples

Ⓐ $\mathbb{R}^n$ with the Euclidean metric is complete.

Ⓑ $C([0,1], \mathbb{R})$ with the metric induced by the sup norm is complete.

Proposition (Prop 4.20 + Thm 4.24)

The following three statements about (M,d) are always true.

(i) (M, τ^d) is compact

$\Updownarrow$ ← Thm (page 1)

(M, d) is seq. compact $\Rightarrow$ (M,d) is complete

(ii) If $\exists\, p \in M$ s.t. the closed ball $\overline{B(p,\varepsilon)}$ is seq. compact $\forall\, \varepsilon > 0$, then (M,d) is complete.

(iii) (M, τ^d) is compact $\iff$ (M,d) is complete AND totally bounded

Totally bounded means that for any given $\varepsilon > 0$, we can cover M by finitely many balls all of the same radius $= \varepsilon$.

Rmks Note that compact $\Leftarrow\!\!\!\Rightarrow$ seq. compact

proof of Thm $(\Leftarrow)$ $\longrightarrow$ $\Downarrow$

totally bounded

& (i) gives compact $\Leftarrow\!\!\!\Rightarrow$ seq. compact

$\Downarrow$

complete

Plus, (ii) $\Rightarrow$ (i) in the sense that (M,d) seq. compact $\Rightarrow$ every <u>closed</u> ball is seq. compact.

proof of (ii) Assume that $\exists\, q \in M$ s.t. $\overline{B(q,\varepsilon)}$ is seq. compact $\forall\, \varepsilon > 0$.

Let (p_n) be a Cauchy sequence in M.

Then for $\varepsilon = 1$, $\exists\, N \in \mathbb{N}$ s.t.

$$n, m \geq N \Rightarrow d(p_n, p_m) < 1$$

Let $R = d(q, p_N)$. Then

$$n \geq N \Rightarrow d(p_n, p) \leq d(p_n, p_N) + d(p_N, q)$$

$$\leq R + 1$$

$$\Rightarrow p_n \in B := \overline{B(q, R+1)}.$$

Since B is seq. compact, $\exists (p_{n_K})$ a subsequence of (p_n) such that $p_{n_K} \to p$ for some $p \in B$. But (p_n) is Cauchy. So, $p_n \to p$.

$\boxtimes$

What's left?

proof of Complete + totally bounded

$\Downarrow$

seq. compact

Assume that (M, d) is complete AND totally bounded. Pick (x_n) any sequence in M. We will construct a subsequence that is Cauchy, hence a subsequence that converges.

We use the same idea as in page 4.

Since (M, d) is totally bounded, we can write $M = \bigcup_{i=1}^{K} B(p_i, 1/2)$ for some

$p_1, \dots, p_K \in M$. In particular, up to relabelling, we may assume that $\exists$ infinitely many terms of (x_n) that lie in $B_1 := B(p_1, 1/2)$.

Let n_1 be the smallest n for which $x_n \in B_1$.

Now, we can also write

$$M = \bigcup_{j=1}^{K'} B(q_j, 1/4) \text{ for some}$$

$q_1, \dots, q_{K'} \in M$. And since a subset of these balls must cover B_1, we may assume that $\exists$ ∞ many elements of (x_n) in $B_2 := B_1 \cap B(q_1, 1/4)$. Again, we let n_2 be the smallest n for which $x_n \in B_2$

and we proceed this way

to construct a subsequence (x_{n_j}) that has the property that given $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t. $\varepsilon > 1/2^{N-1}$ and

$$j, k \geq N \Rightarrow d(x_{n_j}, x_{n_k}) < \varepsilon$$

That is, (x_{n_j}) is Cauchy as we wanted.

In fact, for each $i \in \mathbb{N}$ we construct balls $B(\tilde{p}_i, 1/2^i)$, where $\tilde{p}_1 = p_1$ and $\tilde{p}_2 = q_1$, and nested non-empty closed sets

$$C_i := \mathrm{cl}\left(\bigcap_{j=1}^{i} B(\tilde{p}_j, 1/2^j)\right);$$ and we pick

$x_{n_i} \in C_i$.

Thus,

$$j, k \geq N \Rightarrow x_{n_j} \,\&\, x_{n_k} \in C_N$$

$$\Rightarrow d(x_{n_j}, x_{n_k}) \leq d(x_{n_j}, \tilde{p}_N) + d(\tilde{p}_N, x_{n_k})$$
$$\leq 1/2^N + 1/2^N$$

The completion of a metric space

In general, if $(\hat{M}, \hat{d})$ is a metric space and we can find $\emptyset \neq M \subset \hat{M}$ such that $cl(M) = \hat{M}$ and every Cauchy sequence in M converges in $\hat{M}$, then $(\hat{M}, \hat{d})$ is complete.

Example $(\hat{M}, \hat{d}) = (\mathbb{R}, \text{usual})$ and $M = \mathbb{Q}$.

Definition A completion of a metric space (M, d) is a pair $((\hat{M}, \hat{d}), f)$ where

- $(\hat{M}, \hat{d})$ is complete,
- $f : (M, d) \longrightarrow (\hat{M}, \hat{d})$ is an isometric immersion,
- and $cl(f(M)) = \hat{M}$

Proposition Given (M, d), a completion $((\hat{M}, \hat{d}), f)$ as above always exists (Worksheet #12)