Math 220 - Lecture # 10

Nov 26, 2024

Today: $\begin{cases} \text{Metric topology (again)} \\ \text{Examples} \end{cases}$

Recall (Lecture # 2) that given a metric space (M, d), the metric topology (induced by d) τ^d is defined as follows:

- $\mathcal{U} \subset M$ is open $(\mathcal{U} \in \tau^d) \iff$ given $p \in \mathcal{U}$ we can find $\delta > 0$ such that $B(p, \delta) \subset \mathcal{U}$

RmK.: $\tau^d = \tau^\beta$, where β is the collection of all open balls on (M, d).

Definition / Proposition (Equivalent Metrics)
(Lemma 4.1)

Let M be a set and let d and d' be two metrics on M. Then TFAE:

(i) $\tau^d = \tau^{d'}$

(ii) given $p \in M$ and $\delta > 0$ we can find $r, r' > 0$ such that $B_d(p,r) \subset B_{d'}(p,\delta)$ and $B_{d'}(p,r') \subset B_d(p,\delta)$

If any of these holds, then we say that d and d' are equivalent metrics

proof (i) $\Rightarrow$ (ii) Pick $p \in M$ and $\delta > 0$. Since $\tau^d = \tau^{d'}$, $B_{d'}(p,\delta) \in \tau^d$. Thus, $\exists$ $r > 0$ s.t. $B_d(p,r) \subset B_{d'}(\underset{\underset{p}{\cup}}{p},\delta)$. Changing the roles of d and d' we obtain the other containement

(ii) $\Rightarrow$ (i) It suffices to show that given $p \in M$ and $\delta > 0$ we have that

$$B_{d'}(p,\delta) \in \tau^d \quad \text{and} \quad B_d(p,\delta) \in \tau^{d'}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\tau^{d'} \subset \tau^d \qquad\qquad \tau^d \subset \tau^{d'}$$

Take $q \in B_{d'}(p,\delta)$. Since $B_{d'}(p,\delta) \in \tau^{d'}$, $\exists\, r_q > 0$ s.t. $B_{d'}(q, r_q) \subset B_{d'}(p,\delta)$. Now, by (ii), for this $r_q > 0$, $\exists\, r > 0$ s.t.

$$B_d(q, r) \subset B_{d'}(q, r_q) \subset B_{d'}(p,\delta)$$

Thus, $B_{d'}(p,\delta) \in \tau^d$. Again, changing the roles of d and d' we obtain the other assertion. $\blacksquare$

<u>Rmk:</u> For (ii) $\Rightarrow$ (i), take $\mathcal{U} \in \tau^d$ and $p \in \mathcal{U}$. Then $\exists\, \delta > 0$ s.t. $B_d(p,\delta) \subset \mathcal{U}$. By (ii), for this $\delta > 0$, $\exists\, r' > 0$ s.t. $B_{d'}(p, r') \subset B_d(p,\delta)$. But then since p was arbitrary, this means that $\mathcal{U} \in \tau^{d'}$. Thus, $\tau^d \subset \tau^{d'}$. Again, the argument is symmetric in d & d'.

Example On $M = \mathbb{R}^2$, the following two metrics are equivalent.

- $d(x,y) := \left((x_1 - y_1)^2 + (x_2 - y_2)^2 \right)^{1/2}$
- $d'(x,y) := \max\{ |x_1 - y_1|, |x_2 - y_2| \}$,

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Proposition (A criterion for $\tau^{d} = \tau^{d'}$)

(Exercise 4.1)

If d and d' are two metrics on a set M and $\exists\, A, B \in \mathbb{R}_{>0}$ such that

$$A\, d(x,y) \leq d'(x,y) \leq B\, d(x,y) \qquad \forall\, x \neq y \in M$$

then d and d' are equivalent.

Proof By assumption, given $p \in M$ and $\delta > 0$ we have that

$$B_{d'}(p, \underbrace{A\delta}_{:= r'}) \subset B_{d}(p, \delta) \quad \& \quad B_{d}(p, \underbrace{\delta/B}_{:= r}) \subset B_{d'}(p, \delta).$$

This is exactly condition (ii) on page 2. ∎

"Recovering the definitions of convergence and continuity from Analysis."

Proposition

(i) Let (M, d) be a metric space. A sequence (p_n) of points of M converges to a point $p \in M$ (w.r.t. τ^d) if and only if given $\delta > 0$, we can find $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow p_n \in B(p, \delta)$$

(ii) Let (M, d) and (N, d') be two metric spaces. A function $f : (M, \tau^d) \longrightarrow (N, \tau^{d'})$ is continuous if and only if given $p \in M$ and $\varepsilon > 0$, we can find $\delta > 0$ such that

$$f\big(B(p, \delta)\big) \subset B\big(f(p), \varepsilon\big)$$

proof

(i) First, assume that $p_n \to p$. By definition of convergence, given $\delta > 0$, since $B = B(p, \delta) \in \tau^d$ & $p \in B$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow p_n \in B(p, \delta)$.

Conversely, if we pick $\mathcal{U} \in \tau^d$ with $p \in \mathcal{U}$, then $\exists\, \delta > 0$ s.t. $B(p, \delta) \subset \mathcal{U}$. Now, by assumption, for this $\delta > 0$, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow p_n \in B(p, \delta)$$
$$\Rightarrow p_n \in \mathcal{U}$$

which means that $p_n \to p$ since $\mathcal{U} \in \tau^d$ was arbitrary.

(ii) First, assume that f is continuous and pick $p \in M$ and $\varepsilon > 0$. Since $B = B(f(p), \varepsilon) \in \tau^{d'}$, $f^{-1}(B) \in \tau^d$. But then, since $p \in f^{-1}(B)$, $\exists\, \delta > 0$ s.t. $B(p, \delta) \subset f^{-1}(B)$.

Now,

$$B(p, \delta) \subset f^{-1}(B) \Rightarrow f\big(B(p,\delta)\big) \subset \underset{\displaystyle B = B(f(p), \varepsilon).}{\underset{\cap}{f\left(f^{-1}(B)\right)}}$$

Conversely, if we pick $p \in M$ and $V \in \tau^{d'}$ containing $f(p)$, then $\exists\, \varepsilon > 0$ s.t.

$B(f(p),\varepsilon) \subset V$. Moreover, for this $\varepsilon > 0$, by assumption, $\exists\ \delta > 0$ s.t.

$$f(B(p,\delta)) \subset B(f(p),\varepsilon) \subset V.$$

But this means that f is continuous @ p, since $p \in B(p,\delta)$, $B(p,\delta) \in \tau^d$ and $f(B(p,\delta)) \subset V$.

$\blacksquare$

Any metric space is first-countable. In particular, sequential continuity = continuity. (see, e.g., Lecture # 3)

Proposition (= Prop. 4.3). Let (M,d) be a metric space and (Y,τ) a topological space. Then $f: (M,\tau^d) \to (Y,\tau)$ is continuous @ $p \in M$ iff given $p_n \to p$, we have that $f(p_n) \to f(p)$.

Proof

$(\Rightarrow)$ Lecture #3

$(\Leftarrow)$ By contradiction, assume that f is not continuous at $p \in M$. Then $\exists\, V \in \mathcal{T}$ with $f(p) \in V$ for which we cannot find $\mathcal{U} \in \mathcal{T}^d$ with $p \in \mathcal{U}$ satisfying that $f(\mathcal{U}) \subset V$. Since $\mathcal{U} \subset f^{-1}(V) \Rightarrow f(\mathcal{U}) \subset V$, the inexistency of such $\mathcal{U}$s implies that $\nexists\, \delta > 0$ s.t. $B(p, \delta) \subset f^{-1}(V)$ (open balls are in $\mathcal{T}^d$). In particular, for $\delta_n = \frac{1}{n}$, we can find $p_n \in B(p, \delta_n)$ s.t. $f(p_n) \notin V$. But then $p_n \to p$, whereas $f(p_n) \not\to f(p)$.

(We cannot find $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow f(p_n) \in V$)

$\boxtimes$

Sequences on metric spaces

Fix a metric space (M, d).

① If $p_n \to p$ in M and $M \ni q \neq p$, then $p_n \not\to q$.

② If d' is another metric on M s.t. $\tau^d = \tau^{d'}$, then $p_n \to p$ in $(M, d) \iff p_n \to p$ in (M, d').

③ If $p_n \to p$ in M, then (p_n) is bounded.

④ (Lemma 4.5) If $A \subset M$ and $p \in cl(A)$, then $\exists\, p_n \to p$, where (p_n) is a sequence of points of A.

proof Since $p \in cl(A) = A \cup \partial A$, and $\forall n \in \mathbb{N}$, $B_n = B(p, 1/n) \in \tau^d$ & $p \in B_n$, we have that $B_n \cap A \neq \emptyset$. Thus, for each $n \in \mathbb{N}$ we can choose $p_n \in B_n \cap A$ and, by construction, $p_n \to p$. ∎

Some examples of metric spaces.

(A) $M = \mathbb{R}^2$ with

$$d_1(x,y) := \begin{cases} \|x-y\| & \text{if } x = \lambda y \text{ for some } \lambda \in \mathbb{R}, \\ \|x\| + \|y\| & \text{otherwise} \end{cases}$$

(French Metro Metric / Paris Metric)

- Here, $\|\cdot\|$ denotes the usual Euclidean norm.

OR

$$d_2(x,y) := |x_1 - y_1| + |x_2 - y_2|$$

(Manhattan / taxicab / ℓ^1 ...)

OR

$$d_3(x,y) := \begin{cases} 0 & \text{if } x = y \\ \|x\| + \|y\| & \text{if } x \neq y \end{cases}$$

(British rail metric)

Ⓑ $M = B(X, \mathbb{R}) := \{ f : X \to \mathbb{R};\ f \text{ is bounded} \}$

$(X \neq \emptyset)$ with

$$d(f,g) := \sup_{x \in X} \{ |f(x) - g(x)| \}$$

Ⓒ $M = \mathbb{N}$ with

$$d(n,m) = \frac{|m-n|}{mn} \qquad \text{OR}$$

$$\tilde{d}(n,m) = \begin{cases} 0 & \text{if } m = n \\ 1 + 1/(m+n) & \text{if } m \neq n \end{cases}$$

Ⓓ $M = C([0,1], \mathbb{R}) := \{ f : [0,1] \to \mathbb{R};\ f \text{ is continuous} \}$ with

$$d(f,g) := \int_0^1 |f(x) - g(x)| \, dx$$

Exercise : Verify the axioms of a metric space are indeed satisfied.