

Topological groups

Aline Zanardini

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Definition and examples

Definition 1. A **topological group** is a group $(G, \cdot)$ endowed with a topology τ such that both

- the group operation $\cdot : G \times G \rightarrow G, (g, h) \mapsto g \cdot h$ and
- the inversion $G \rightarrow G, g \mapsto g^{-1}$

are continuous functions, where on $G \times G$ we consider the product topology induced by τ .

Example 1. Every group is a topological group when endowed with the indiscrete topology.

Example 2. Every group is a topological group when endowed with the discrete topology.

Example 3. $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$ are topological groups with the usual topology.

Example 4. The group $GL(n, \mathbb{R})$ is a topological group, where we view it as a subspace of the space of all square matrices $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ (with the usual topology).

Example 5.

- (i) If G is a topological group and $H \leq G$ is a subgroup, then H is a topological group with the subspace topology.
 - In particular, $SL(n, \mathbb{R}), O(n, \mathbb{R}), SO(n, \mathbb{R}), \dots$ are all topological groups.
- (ii) If G is a topological group and $N \trianglelefteq G$ is a normal subgroup, then G/N is a topological group with the quotient topology.
- (iii) If G and G' are topological groups, then $G \times G'$ is a topological group with the product topology.

The fundamental group

Definition 2. If $(G, \cdot)$ is a topological group, then one can define a new “multiplication” of paths $\gamma, \gamma' : [0, 1] \rightarrow G$ by setting

$$(\gamma * \gamma')(s) = \gamma(s) \cdot \gamma'(s) \quad \forall s \in [0, 1]$$

Note that if γ and γ' are loops based at the neutral element $1_G \in G$, then $\gamma * \gamma'$ is also a loop based at 1_G . Moreover, if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$ in $\pi_1(G, 1_G)$ then $[\alpha * \beta] = [\alpha' * \beta']$. That is, the operation $*$ between paths on G induces a well defined operation on the set of homotopy classes $\pi_1(G, 1_G)$.

Problems

Prove the following claims are indeed true.

Claim 1. If G is a topological group and $H \leq G$ is a subgroup, then one can endow G/H with the quotient topology induced by the natural surjection $G \rightarrow G/H$. The space of left cosets G/H is Hausdorff if and only if $H \subset G$ is closed. Moreover, the action $G \times G/H \rightarrow G/H$ given by left multiplication is a continuous function, where on $G \times G/H$ we consider the product topology.

Claim 2. If G and G' are topological groups, then a homomorphism of groups $\varphi : G \rightarrow G'$ is continuous if and only if it is continuous at the neutral element $1_G \in G$.

Claim 3. If (G, τ) is a topological group and $H \leq G$ is a subgroup, then $H \in \tau$ implies $G \setminus H \in \tau$. Moreover, if G is finite, then the converse also holds. That is, if $H \subset G$ is closed, then H is open.

Claim 4. The first isomorphism theorem fails for topological groups. In fact, if we consider $(\mathbb{R}, \tau_{usual})$ and $(\mathbb{R}, \tau_{discrete})$, then the identity

$$id : (\mathbb{R}, \tau_{discrete}) \rightarrow (\mathbb{R}, \tau_{usual})$$

is a continuous homomorphism with a trivial kernel, but $(\mathbb{R}, \tau_{discrete})$ and $(\mathbb{R}, \tau_{usual})$ are not isomorphic as topological groups.

Claim 5. If G is topological group and G_0 denotes the connected component of the neutral element $1_G \in G$, then $G_0 \trianglelefteq G$ is a closed normal subgroup. In fact G/G_0 is the group of connected components of G .

Claim 6. The operation $*$ from Definition 2 is always commutative and it agrees with the operation $\star$ we had defined previously. In particular, if G is a topological group, then $\pi_1(G, 1_G)$ is always Abelian.