The spectrum of a ring

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1 Rings and ideals

Definition 1.1. *A* <u>*ring*</u> *(with unit) is a set A endowed with two binary operations $(a, b) \mapsto a + b$ and $(a, b) \mapsto a \cdot b$ which satisfy the following axioms:*

- $(A, +)$ *is an abelian group*
- $(A, \cdot)$ *is a monoid:*
  - *(associativity)* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
  - *(unity) there exists $1_A \in A$ such that $1_A \cdot a = a = a \cdot 1_A$ for all $a \in A$*
- *The "multiplication" is distributive with respect to the "addition":*
  - $a \cdot (b + c) = a \cdot b + a \cdot c$
  - $(b + c) \cdot a = b \cdot a + c \cdot a$

We say a ring $(A, +, \cdot)$ is commutative if the "multiplication" is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in A$.

Definition 1.2. *Let $(A, +, \cdot)$ be a commutative ring. We say $I \subset A$ is an* <u>*ideal*</u> *of A, denoted by $I \trianglelefteq A$, if I is a subgroup of $(A, +)$ and for all $a \in A$ the set $aI \doteq \{a \cdot u\,;\, u \in I\}$ is contained in I. That is, given $u, v \in I$ and $a \in A$ the elements $u + v$ and $a \cdot u$ are in I* [1]*.*

Definition 1.3. *Let $(A, +, \cdot)$ be a commutative ring and let $I \trianglelefteq A$ with $I \neq A$. We say I is* <u>*prime*</u> *if given $a, b \in A$ such that $a \cdot b \in I$ we have that either $a \in I$ or $b \in I$.*

Definition 1.4. *Let $(A, +, \cdot)$ and $(B, \oplus, \times)$*[2] *be two rings. We say a function $\varphi : A \to B$ between the two underlying sets is a morphism of rings if $\varphi(1_A) = 1_B$ and for all $a, a' \in A$ we have*

- $\varphi(a + a') = \varphi(a) \oplus \varphi(a')$
- $\varphi(a \cdot a') = \varphi(a) \times \varphi(a')$

[1] If A is not commutative then any such I is called a left ideal.

[2] Here $\oplus$ denotes the "addition" on B and $\times$ the "multiplication"

2 Spec(A)

Let $(A, +, \cdot)$ be a commutative ring. The spectrum of A (as a set), denoted by Spec(A) consists in the set of all prime ideals of A:

$$\mathrm{Spec}(A) = \{\mathfrak{p} \trianglelefteq A\,;\, \mathfrak{p} \text{ is prime}\}$$

Example 2.1. *The points in Spec($\mathbb{Z}$) are the ideals (0) and $p\mathbb{Z}$, where p is a prime number.*

Example 2.2. *The points in Spec($\mathbb{C}[x]$) are the ideals (0) and the ideals $(x - \lambda)$, where $\lambda \in \mathbb{C}$*

Now, given A we can construct a topology on Spec(A) in the following way:

- If $I \trianglelefteq A$ is an ideal we let $V(I) \doteq \{\mathfrak{p} \in \mathrm{Spec}(A)\,;\, \mathfrak{p} \supset I\}$,
- and we declare that the closed subsets in Spec(A) are the subsets of the form $V(I)$ for some $I \trianglelefteq A$.

This gives a topology because

- $V((0)) = \mathrm{Spec}(A)$ and $V(A) = \emptyset$
- $V(I) \cup V(J) = V(IJ)$ where IJ is the ideal generated by finite sums of products of the form $u \cdot v$ with $u \in I$ and $v \in J$
- $V(I) \cap V(J) = V(I + J)$ where $I + J = \{u + v\,;\, u \in I, v \in J\}$ is the smallest ideal of A which contains both I and J [3]

And it turns out that this topology is the topology generated by the collection of open subsets of the form

$$D_a \doteq \{\mathfrak{p} \in \mathrm{Spec}(A)\,;\, a \notin \mathfrak{p}\}$$

where $a \in A$.

Problem 2.3. *If $\varphi : A \to B$ is a morphism of rings then you can check the following:*

(i) If $\mathfrak{p} \in$ Spec(B) then $\varphi^{-1}(\mathfrak{p}) \in$ Spec(A)

(ii) The function Spec(B) $\to$ Spec(A) given by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ is continuous (w.r.t. the topologies we defined above)

Problem 2.4. *If $S \subset$ Spec($\mathbb{Z}$) is infinite, then you can check that S is closed if and only if $S =$ Spec($\mathbb{Z}$). This tells us that the topology in Spec($\mathbb{Z}$) we defined above looks like the cofinite topology on the set of prime numbers, except that on Spec($\mathbb{Z}$) the closure of (0) is the whole space.*

[3] In fact given an arbitrary collection of ideals $\{I_\alpha\}$ of A we have that $\bigcap V(I_\alpha) = V\left(\sum I_\alpha\right)$ where $\sum I_\alpha$ is defined to be the smallest ideal of A containing all the I_α.