Borel σ-algebra

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1 σ-algebras

Definition 1.1. *Let X be a set. A σ-algebra on X is a non-empty collection $\mathcal{C}$ of subsets of X such that*

(i) $\mathcal{C}$ is closed under countable unions and

(ii) $\mathcal{C}$ is closed under taking complements

In particular, if $\mathcal{C}$ is a σ-algebra on X, then $\mathcal{C}$ is also closed under countable intersections and we always have $\emptyset, X \in \mathcal{C}$.

σ-algebras vs. topologies Note that the axioms above are very similar to those defining what a topology is. However, topologies are closed under arbitrary unions and only closed under finite intersections. Moreover, topologies are not necessarily closed under taking complements – otherwise all open subsets would also be closed subsets.

Nonetheless, any topological space comes equipped with the σ-algebra generated by the collection of open subsets, i.e., the topology.

1.1 The Borel σ-algebra

In general, given a set X and a collection of subsets of X, say $\mathcal{A}$, one can consider the smallest σ-algebra on X which contains $\mathcal{A}$, denoted by $\mathcal{C}(\mathcal{A})$. This σ-algebra $\mathcal{C}(\mathcal{A})$ is given by the intersection of all σ-algebras $\mathcal{C}$ on X such that $\mathcal{A} \subset \mathcal{C}$.

Definition 1.2. *If (X, τ) is a topological space, then the σ-algebra $\mathcal{C}(\tau)$ is called the Borel σ-algebra of (X, τ) – the smallest σ-algebra on X which contains the topology τ. Any subset of X that lies in $\mathcal{C}(\tau)$ is called a Borel subset.*

Example 1.3. *On $(\mathbb{R}, \tau_{usual})$, the corresponding Borel σ-algebra is the smallest σ-algebra on X which contains all the intervals (since they generate the usual topology).*

Question 1.4. *Let (X, τ) be a topological space, and consider its Borel σ-algebra $\mathcal{C}(\tau)$. What is the smallest topology on X that contains $\mathcal{C}(\tau)$?*

Remark 1.5. *If $\mathcal{C}$ is a σ-algebra on X which is generated by a countable collection $\mathcal{A}$ of subsets of X, then letting $\mathcal{B}$ be the collection of all finite intersections of elements of $\mathcal{A}$ we can show that $\mathcal{B}$ is a basis for a topology on X and, moreover, $\mathcal{C} = \mathcal{C}(\tau^{\mathcal{B}})$, where $\tau^{\mathcal{B}}$ denotes the topology generated by the collection $\mathcal{B}$.*