

Analysis III - 203(d)

Winter Semester 2024

Session 6: October 17, 2024

Exercise 1 We have the vector field

$$\vec{F}(x_1, x_2) = (x_1 x_2, x_2^2).$$

and the domains

$$\begin{aligned}\Omega_1 &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}, \\ \Omega_2 &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1, x_1 > 0\},\end{aligned}$$

Verify Green's theorem for the vector field $\vec{F}$ with the domains Ω_1 and Ω_2 . You need to find parameterizations of the boundary first.

Exercise 2 Consider the triangle domain

$$T := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}.$$

and the vector field

$$\vec{F}(x_1, x_2) = \left(x_1 x_2 + \frac{x_1}{1 + x_1^2 + x_2^2}, x_1 x_2 + \frac{x_2}{1 + x_1^2 + x_2^2} \right)$$

Find the curve integral of $\vec{F}$ along the boundary of T using Green's theorem.

Exercise 3 Consider the parabolic arc

$$\Gamma := \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 < x_1 < 1, x_2 = 3(1 - x_1^2)\}.$$

Find the curve integral $\int_{\Gamma} \vec{F} \cdot \vec{n} \, dl$, where

$$\vec{F}(x_1, x_2) = (x_1(2 - \cos(x_1 x_2))^2, x_2(2 + \cos(x_1 x_2))^2)$$

and where $\vec{n}$ is the unit vector along Γ , perpendicular to Γ and having non-negative x_2 component.

Exercise 4 Find the tangential vector $\dot{\gamma}(t)$, the unit tangential vector $\vec{\tau}$ and the unit normal $\vec{n}$ of the simple closed curve

$$\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(t)(1 + \sin(2t)^2)).$$

Find the values of γ and $\vec{\tau}$ for a few values of $t \in [0, 2\pi]$, such as $t = \frac{\pi}{4}, 2\frac{\pi}{4}, 3\frac{\pi}{4}, \dots, 7\frac{\pi}{4}$

Exercise 5 Find the area of the graph of ϕ over $\Omega = [0, 1] \times [0, 1]$, where

$$\phi(s, t) := \sqrt{s^2 + t^2}$$

Find the integral of the function

$$f(x_1, x_2, x_3) := x_1 x_2 x_3$$

over the graph of ϕ over Ω .

Exercise 6 The parameterization

$$\Phi : [0, 2\pi) \times (0, 1) \rightarrow \mathbb{R}^3, \quad (\theta, z) \mapsto ((1+z)\cos(\theta), (1+z)\sin(\theta), z)$$

describes a surface S . Find the surface area of S .

Exercise 7 Let $f(x_1, x_2, x_3) = x_1 x_2 + x_3^2$ and consider the surface

$$S := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 < x_3 < 1, x_1^2 + x_2^2 = x_3^2 \}$$

Find a parameterization of S and compute the surface integral $\iint_S f \, d\sigma$.