18. 12. 2024

Poisson Problem over $\mathbb{R}$ (cont.)

19. 12. 2024

Integral Equations

III Application to the Poisson problem

For the Poisson problem

$$-u''(x) + k^2 u(x) = f(x)$$

we have seen

$$\mathcal{F}[u](\alpha) = \frac{1}{k^2+\alpha^2} \cdot \mathcal{F}[f](\alpha)$$

We know from the FT table

$$\frac{e^{-k|x|}}{k} \quad \text{has the FT} \quad \sqrt{\frac{2}{\pi}}\, \frac{1}{k^2+\alpha^2}$$

Hence

$$g(x) = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{k} e^{-k|x|} \implies \mathcal{F}[g](\alpha) = \frac{1}{k^2+\alpha^2}$$

We can write

$$\mathcal{F}[u](\alpha) = \frac{1}{k^2+\alpha^2}\mathcal{F}[f](\alpha) = \Big[\mathcal{F}[g]\cdot\mathcal{F}[f]\Big](\alpha)$$

Using the convolution theorem:

$$u(x) = \frac{1}{\sqrt{2\pi}}(g * f)(x)$$

Explicitly, with our particular choice of g:

$$u(x) = \frac{1}{2k}\left(e^{-k|\cdot|} * f(\cdot)\right)(x)$$

$$= \frac{1}{2k}\int_{-\infty}^{+\infty} f(y)\, e^{-k|x-y|}\, dy$$

- This is a general solution formula for

$$-u''(x) + k^2 u(x) = f(x)$$

over the real line. It is not necessarily the only solution but it will satisfy

$$\lim_{x \to \pm\infty} u(x) = 0$$

- Major difficulty: computing the convolution.

- Similar techniques are possible for many other differential equations over $\mathbb{R}$.

IV Examples

(1) Consider the source term

$$f(x) = e^{-|x|}$$

Using our solution formula, the differential equation

$$-u''(x) + k^2 u(x) = f(x)$$

has the solution

$$u(x) = \frac{1}{2k} \int_{-\infty}^{+\infty} e^{-|y|} e^{-k|x-y|} \, dy$$

To compute the integral, we split it up and use case distinctions.

We begin with

$$u(x) = \frac{1}{2k} \int_{-\infty}^{0} e^{y} e^{-k|x-y|} \, dy + \int_{0}^{\infty} e^{-y} e^{-k|x-y|} \, dy$$

Because of $|x-y|$ we make a case distincion.

Consider the case $x \geq 0$.

$$u(x) = \frac{1}{2k} \int_{-\infty}^{0} e^{y} e^{-k(x-y)} \, dy + \frac{1}{2k} \int_{0}^{x} e^{-y} e^{-k(x-y)} \, dy$$

$$+ \frac{1}{2k} \int_{x}^{\infty} e^{-y} e^{-k(y-x)} \, dy$$

$$= \frac{e^{-kx}}{2k} \int_{-\infty}^{0} e^{(k+1)y} \, dy + \frac{e^{-kx}}{2k} \int_{0}^{x} e^{(k-1)y} \, dy + \frac{e^{kx}}{2k} \int_{x}^{\infty} e^{-(k+1)y} \, dy$$

$$= \frac{e^{-kx}}{2k} \frac{1}{k+1} + \frac{e^{-kx}}{2k} \int_0^x e^{(k-1)y}\,dy + \underbrace{\frac{e^{kx}}{2k} \frac{e^{-(k+1)x}}{k+1}}_{\frac{e^{-x}}{2k(k+1)}}$$

Still assuming $x \geq 0$, we make another case distinction.

If $k=1$, then

$$u(x) = \frac{e^{-x}}{4} + \frac{e^{-x}}{2} \cdot x + \frac{e^{-x}}{4}$$

$$= (1+x)\frac{e^{-x}}{2}$$

We check the differential equation

$$u'(x) = \frac{e^{-x}}{2} - (1+x)\frac{e^{-x}}{2} = (-x)\frac{e^{-x}}{2}$$

$$u''(x) = -\frac{e^{-x}}{2} - (-x)\frac{e^{-x}}{2} = (x-1)\frac{e^{-x}}{2}$$

Plugging this in, we check easily

$$-u''(x) + u(x) = f(x), \qquad x \geq 0$$

(Exercise)

Next, still assuming $x \geq 0$, we handle the case $k \neq 1$.

$$u(x) = \frac{e^{-kx}}{2k(k+1)} + \underbrace{\frac{e^{-kx}}{2k} \cdot \frac{e^{(k-1)x} - 1}{k-1}}_{\dfrac{e^{-x} - e^{-kx}}{2k(k-1)}} + \frac{e^{-x}}{2k(k+1)}$$

We split the middle term

$$u(x) = \frac{e^{-kx}}{2k(k+1)} - \frac{e^{-kx}}{2k(k-1)} + \frac{e^{-x}}{2k(k-1)} + \frac{e^{-x}}{2k(k+1)}$$

$$= \frac{e^{-kx}(k-1)}{2k(k^2-1)} - \frac{e^{-kx}(k+1)}{2k(k^2-1)} + \frac{e^{-x}(k-1)}{2k(k^2-1)} + \frac{e^{-x}(k+1)}{2k(k^2-1)}$$

$$= -\frac{e^{-kx}}{k(k^2-1)} + \frac{e^{-x}}{(k^2-1)}$$

One checks that this satisfies

$$-u''(x) + k^2 u(x) = f(x), \qquad x \geq 0$$

That completes the case $x \geq 0$.
It remains to discuss the case $x \leq 0$

Instead of repeating the same computations,
we take a short cut.
Recall that we study the proposed solution

$$u(x) = \frac{1}{2k} \int_{-\infty}^{+\infty} e^{-|y|} e^{-k|x-y|} \, dy$$

Suppose that $x = -w$ for some positive $w > 0$. Then

$$u(x) = \frac{1}{2k}\int_{-\infty}^{\infty} e^{-|y|} e^{-k|-w-y|} dy \qquad z = -y$$

$$= \frac{1}{2k}\int_{-\infty}^{+\infty} e^{-|z|} e^{-k|-w+z|} dz$$

$$= \frac{1}{2k}\int_{-\infty}^{+\infty} e^{-|z|} e^{-k|w-z|} dz$$

$$= u(w) = u(-x)$$

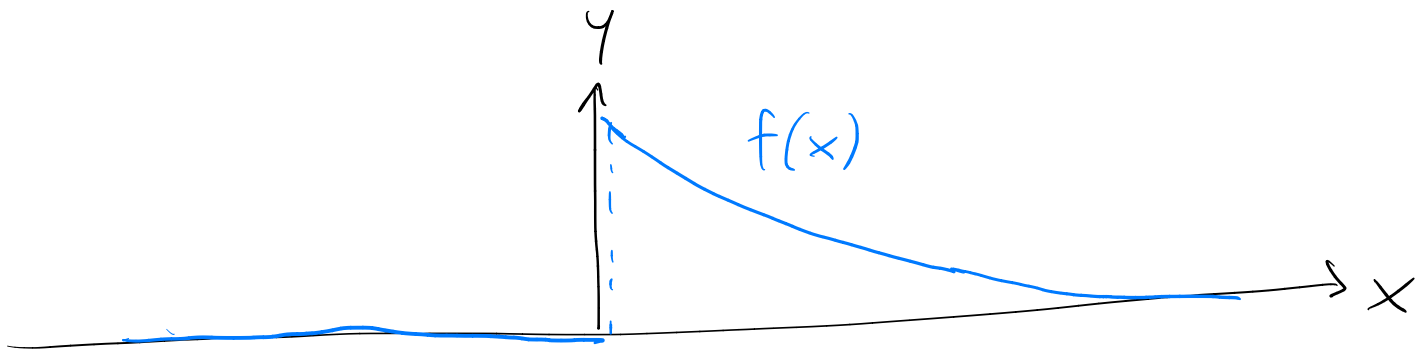
That means that u is an even function.

Since u satisfies the differential equation for $x > 0$, and so it must satisfy the differential equation for $x < 0$.

Lastly, we check that $u(x)$ in this form is continuous at $x=0$, and so is its derivative.

② **Example:** Consider the source term

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



The solution formula provides

$$u(x) = \frac{1}{2k} \int_0^\infty e^{-y} e^{-k|x-y|} dy$$

Here, the integral bounds simplify because $f(x) = 0$ for negative $x < 0$.

To study this, we take a look at the case $x < 0$ first. Then $|x-y| = y - x$. for $y \geq 0$. Hence

$$u(x) = \frac{1}{2k} \int_0^{+\infty} e^{-y-k(y-x)} dy$$

$$= \frac{e^{kx}}{2k} \int_0^{+\infty} e^{-(k+1)y} dy = \frac{1}{2k} \cdot \frac{e^{kx}}{k+1}$$

We then check that for $x < 0$:

$$-u''(x) + k^2 u(x) = k^2 \frac{e^{kx}}{2k(k+1)} - k^2 \frac{e^{kx}}{2k(k+1)} = 0$$

as desired. So the differential equation is satisfied for $x<0$.

For $x > 0$, the integrand switches its form again at $y = x$ and we need a case distinction for $k=1$ and $k \neq 1$.

Details are not provided here. See exam review.

Integral Equations

We study integral equations of convolutional form

$$u(x) + \lambda \int_{-\infty}^{+\infty} K(x-y)\, u(y)\, dy = g(x)$$

While differential equations involve derivatives, integral equations involve integrals.

We will use Fourier transforms to solve them.

Here, $\lambda > 0$ is a positive parameter, K is called the integration kernel, and g is called the source term.

This equation can be written

$$u(x) + \lambda (K * u)(x) = g(x)$$

To find the unknown function u, we use the Fourier transform

$$\hat{u}(\alpha) + \sqrt{2\pi} \cdot \lambda \cdot \hat{K}(\alpha) \cdot \hat{u}(\alpha) = \hat{g}(\alpha)$$

We isolate $\hat{u}(\alpha)$:

$$\left(1 + \sqrt{2\pi}\, \lambda \cdot \hat{K}(\alpha)\right) \hat{u}(\alpha) = \hat{g}(\alpha)$$

$$\hat{u}(\alpha) = \left(1 + \sqrt{2\pi}\, \lambda \hat{K}(\alpha)\right)^{-1} \hat{g}(\alpha)$$

We can find u, if we are able to take the inverse Fourier transform

In the case that $h: \mathbb{R} \to \mathbb{R}$ such that

$$\hat{h}(\alpha) = \left(1 + \sqrt{2\pi}\,\lambda\,\hat{K}(\alpha)\right)^{-1}$$

we have

$$\hat{u}(\alpha) = \hat{h}(\alpha) \cdot \hat{g}(\alpha)$$

Applying the inverse Fourier transform,

$$u(x) = \frac{1}{\sqrt{2\pi}}(h * g)(x)$$

Indeed

$$\hat{u}(\alpha) = \frac{1}{\sqrt{2\pi}}\left(\widehat{h * g}\right)(\alpha)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\,\hat{h}(\alpha) \cdot \hat{g}(\alpha) = \hat{h}(\alpha)\,\hat{g}(\alpha)$$

Example We want the solution $u : \mathbb{R} \to \mathbb{R}$ of the integral equation

$$u(x) + 3 \int_{-\infty}^{+\infty} e^{-|\xi|} u(x - \xi) \, d\xi = e^{-|x|}$$

Here,

$$\lambda = 3, \qquad K(x) = e^{-|x|}, \qquad g(x) = e^{-|x|}$$

The relevant Fourier transforms are

$$\hat{K}(\alpha) = \hat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{2}{1 + \alpha^2}$$

The convolution equation reads:

$$u(x) + 3(K * u)(x) = g(x)$$

FT:

$$\hat{u}(\alpha) + 3\sqrt{2\pi}\,(\hat{K}(\alpha) \cdot \hat{u}(\alpha)) = \hat{g}(\alpha)$$

We isolate $\hat{u}(\alpha)$:

$$\hat{u}(\alpha) = \frac{1}{1 + 3\sqrt{2\pi}\,\hat{K}(\alpha)}\,\hat{g}(\alpha)$$

We simplify the last expression:

$$\frac{\hat{g}(\alpha)}{1 + 3\sqrt{2\pi}\,\hat{k}(\alpha)} = \frac{\frac{1}{\sqrt{2\pi}}\frac{2}{1+\alpha^2}}{1 + 3\cancel{\sqrt{2\pi}}\,\frac{1}{\cancel{\sqrt{2\pi}}}\frac{2}{1+\alpha^2}}$$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{1/1+\alpha^2}{1 + 6\,\frac{1}{1+\alpha^2}}$$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{1}{1+\alpha^2 + 6}$$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{1}{7+\alpha^2}$$

One checks that the last expression equals the FT of

$$u(x) = \frac{1}{\sqrt{7}} e^{-\sqrt{7}\cdot|x|}$$

That solves the convolutional integral equation.

Remarks on convolutional integral equations:

- applications in physics, engineering, signal processing
- Typically, the inverse Fourier transform, possibly involving the convolution theorem, is the most complicated
- If the denominator $1 + \lambda\sqrt{2\pi}\,\hat{K}(\alpha)$ is zero for some frequency α, then the theory still applies in some circumstances but becomes more complicated.

Review

- Lines and surfaces can be given by equations that define sets:

  Example: $\{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 3, \quad z \geq 0 \}$

  Lines, surfaces, domains

- Parametrizations of lines and surfaces

  Lines: tangent vector

  Domains in 2D: unit tangent and outward pointing unit normal

  Surfaces in 3D: unit tangent and normal along surface

  Domains in 3D: outward pointing unit normal

- Line integrals of scalars $\int_\gamma f \, dl$ and vectors $\int_\gamma \vec{F} \, dl$
- Surface integrals of scalars $\int_S f \, d\sigma$ and vectors $\int_S \vec{F} \, d\sigma$
- The line/surface integrals depend on the parameterization: $\pm$
- If a surface has a boundary, the parametrization of the surface gives a parametrization of the boundary line: $\pm$
- Integral theorems:

2D: Gauss & Green: $\iint_\Omega \operatorname{div} \vec{F} = \int_{\partial\Omega} \vec{F} \cdot \vec{n}$, $\iint_\Omega \operatorname{curl} \vec{F} = \int_{\partial\Omega} \vec{F} \cdot \vec{J}$

3D: Gauss / divergence: $\iiint_\Omega \operatorname{div} \vec{F} = \iint_{\partial\Omega} \vec{F} \cdot \vec{n}$

Stokes: $\iint_S \operatorname{curl} \vec{F} = \int_{\partial S} \vec{F} \cdot \vec{J}$

- Relevant differential operators: gradient, divergence, Laplacian, curl 2D/3D

- Potentials: existence of scalar potential for some vector field?

$$\text{grad } f = \vec{F} \ ?$$

necessary: curl $\vec{F} = 0$

sufficient: " " " " and domain is simply-connected

or the integral of $\vec{F}$ along any line around a single hole equals zero.

IF a potential exists THEN you can compute via a line integral

Distribution theory

- $\mathcal{D}$ is the vector space of smooth functions with compact support
- $\mathcal{D}'$ is the space of distributions
- A distribution is a linear function $T: \mathcal{D} \to \mathbb{R}$ satisfying:

$$\forall\ [a,b] \subseteq \mathbb{R} :\ \exists\ C > 0,\ k \in \mathbb{N}_0 :\ \forall \varphi \in \mathcal{D},\ \operatorname{supp}(\varphi) \subseteq [a,b] :\quad |f(\varphi)| \leq C \sum_{i=0}^{k} \max_{x \in \mathbb{R}} |\partial^i \varphi(x)|$$

- Examples: Dirac delta, Dirac comb,
  $T_f(\varphi) := f(\varphi) := \int f(x)\, \varphi(x)\, dx$, f (locally) integrable

- Distributional derivative: $(\partial_x T)(\varphi) = -T(\partial_x \varphi)$

- Important special case: distributional derivative of piecewise differentiable functions: piecewise derivatives + Dirac Delta at jumps

Fourier series:

standard form $\quad Ff(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)$

complex form $\quad Ff(x) = \sum_{n\in\mathbb{Z}} c_n e^{-\frac{2\pi n}{T} x i}$

transition between coefficients $(a_n, b_n) \longleftrightarrow (c_n, c_{-n})$

Calculation: computing integrals, recognize when to use integration-by-parts

When f is even/odd, then the Fourier series is only a cosine/sine series, and coefficient formula simplifies (see exercise)

- Dirichlet theorem: when does the Fourier series converge? If yes, to what?

- Parseval theorem

Background: orthogonality properties of sine/cosine modes

Fourier transform

- Definition of FT : $\mathcal{F}[f](\alpha) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int f(x)\, e^{-i\alpha x}\, dx$
- Inverse FT $\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha$

- Linearity
- Déculage / modulation
- Plancherel
- Interaction with derivatives

$$\mathcal{F}[f'](\alpha) = i\alpha\, \mathcal{F}[f](\alpha)$$

- Interaction with convolution :

$$\mathcal{F}[f * g] = \sqrt{2\pi}\, \mathcal{F}[f] \cdot \mathcal{F}[g]$$

$$\mathcal{F}[f \cdot g] = \sqrt{2\pi}\, \mathcal{F}[f] * \mathcal{F}[g]$$

- Reminder: the convolution of $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

$$(f * g)(x) = \int f(x-y)\, g(y)\, dy = \int f(y)\, g(x-y)\, dy$$

is linear in f and g, is associative and commutative.

Applications:

- Poisson problem $-u''(x) + k^2 u(x) = f(x)$ over an interval $a < x < b$ using Fourier series

  - Dirichlet boundary conditions: $u(a) = g_a$, $u(b) = g_b$

    split problem into $-u'' + k^2 u = 0$, $u(a) = g_a$, $u(b) = g_b$ and $-u'' + k^2 u = f$, $u(a) = 0$, $u(b) = 0$ and solve separately

    homogeneous source term   homogeneous BC

    Take odd extension of f, develop Fourier series, relate Fourier coefficients of f with Fourier coefficients of u

  - Neumann BC: $u'(a) = 0$, $u'(b) = 0$ use cosine series

    periodic BC: $u(a) = u(b)$ use full Fourier series

  - Extension to general differential equations

    Express the right-hand side as a Fourier series such that the BC are automatically satisfied (sine, cosine, full)

- Poisson problem $-u''(x) + k^2 u(x) = f(x)$ over $\mathbb{R}$ using Fourier transform

$$\Rightarrow \quad -(i\alpha)^2 \hat{u}(\alpha) + k^2 \hat{u}(\alpha) = \hat{f}(\alpha) \qquad \text{(FT equation)}$$

$$\Rightarrow \quad \hat{u}(\alpha) = (k^2 + \alpha^2)^{-1} \cdot \hat{f}(\alpha) \qquad (\text{isolate } \hat{u})$$

$$\Rightarrow \quad u(x) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left[(k^2 + \alpha^2)^{-1}\right] * f(\alpha) \qquad \text{(inverse FT)}$$

Knowing how to compute convolutions

- Convolutional integral equation: $u + \lambda(K * u) = g$

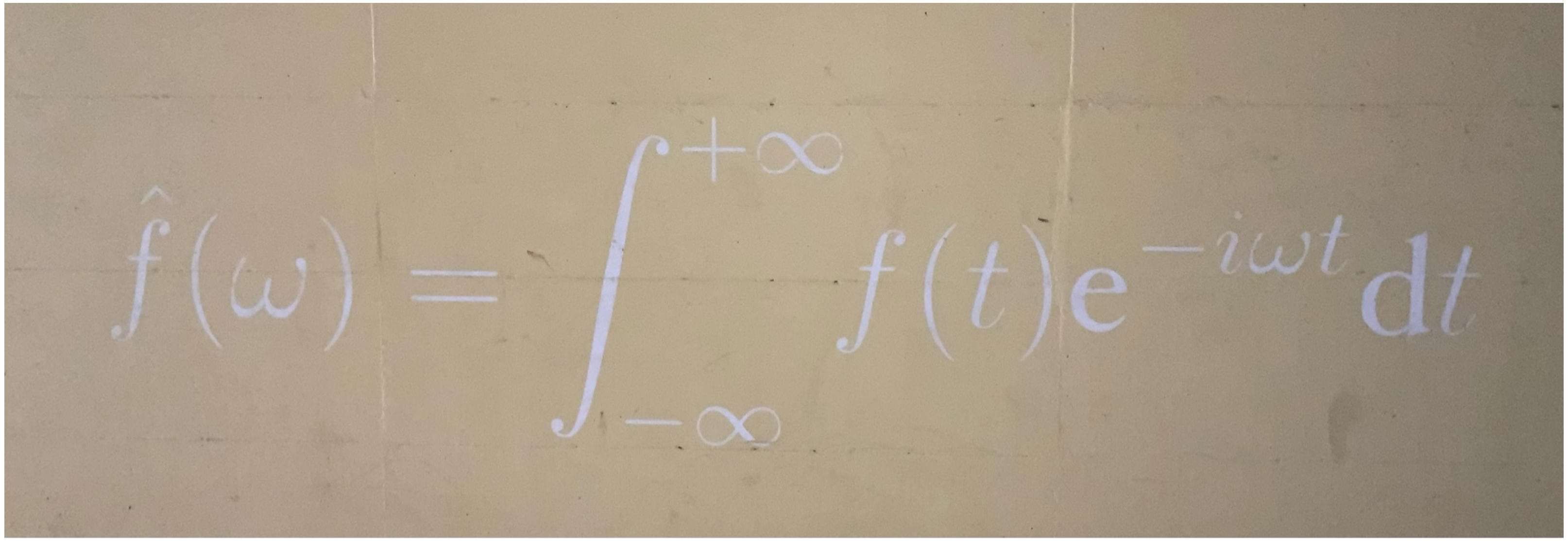
$$\Rightarrow \quad \hat{u} + \lambda \hat{K} * \hat{u} = \hat{g} \qquad \text{(FT equation)}$$

$$\Rightarrow \quad \hat{u} = (1 + \lambda \hat{K})^{-1} \cdot \hat{g} \qquad (\text{isolate } \hat{u})$$

$$\Rightarrow \quad \hat{u} = \mathcal{F}^{-1}\left[(1 + \lambda \hat{K})^{-1} \hat{g}\right] = \mathcal{F}^{-1}\left[(1 + \lambda \hat{K})^{-1}\right] * g \qquad \text{(inverse FT)}$$

Knowing how to find inverse FT or how to compute convolutions

There are many different conventions for the definition of the Fourier transform, without any overwhelming consensus.



Good Luck

for your Exam!