Analysis III - 203(d)

Winter Semester 2024

Session 10: November 21, 2024

Exercise 1

$$ReLU(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \end{cases}$$

$$ReLU^{k}(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^{k} & \text{if } 0 \le x \end{cases} \text{ where } k \in \mathbb{N}_{0}$$

$$f(x) = \begin{cases} -x - 2 & \text{if } -2 < x < -1 \\ -x^{2} - 2x - 1 & \text{if } -1 < x < 0 \\ x^{2} - 2x + 1 & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$j(x) = \begin{cases} x & \text{if } x < 0 \\ 1 + x & \text{if } 0 \le x \end{cases}.$$

Here, employ the definition of distributional derivatives and check against the formula for piecewise differentiable functions as obtained during lecture.

Solution 1 • For the ReLU function, we have for any $\phi \in \mathcal{D}(\mathbb{R})$,

$$\langle ReLU', \phi \rangle = -\langle ReLU, \phi' \rangle = -\int_{-\infty}^{\infty} ReLU(x)\phi'(x)dx = -\int_{0}^{\infty} x\phi'(x)dx = \int_{0}^{\infty} \phi(x)dx = \langle H, \phi \rangle,$$

where H is the Heaviside step function. Thus, we conclude that ReLU' = H in the sense of distributions.

• For k > 1, we find for any $\phi \in \mathcal{D}(\mathbb{R})$,

$$\langle (ReLU^{k})', \phi \rangle = -\langle ReLU^{k}, \phi' \rangle = -\int_{-\infty}^{\infty} ReLU^{k}(x)\phi'(x)dx$$
$$= -\int_{0}^{\infty} x^{k}\phi'(x)dx = \int_{0}^{\infty} kx^{k-1}\phi(x)dx = \langle kReLU^{k-1}, \phi \rangle.$$

Therefore, we conclude that $(ReLU^k)' = kReLU^{k-1}$ in the sense of distributions.

• For the function f, we have for any $\phi \in \mathcal{D}(\mathbb{R})$,

$$\begin{split} \langle f', \phi \rangle &= -\langle f, \phi' \rangle = \int_{-2}^{-1} (x+2) \phi'(x) dx \\ &+ \int_{-1}^{0} (x^2 + 2x + 1) \phi'(x) dx \\ &- \int_{0}^{1} (x^2 - 2x + 1) \phi'(x) dx \\ &+ \int_{1}^{2} (x-2) \phi'(x) dx \\ &= \phi(-1) - \int_{-2}^{-1} \phi(x) dx \\ &+ \phi(0) - \int_{-1}^{0} 2(x+1) \phi(x) dx \\ &+ \phi(0) + \int_{0}^{1} 2(x-1) \phi(x) dx \\ &+ \phi(1) - \int_{1}^{2} \phi(x) dx \\ &= \int_{-2}^{-1} (-1) \phi(x) dx + \int_{-1}^{0} (-2(x+1)) \phi(x) dx \\ &+ \int_{0}^{1} 2(x-1) \phi(x) dx + \int_{1}^{2} (-1) \phi(x) dx + \phi(-1) + 2\phi(0) + \phi(1) \\ &= \langle g + \delta_{-1} + 2\delta_{0} + \delta_{1}, \phi \rangle, \end{split}$$

where the function $g: \mathbb{R} \to \mathbb{R}$ is defined as

$$g(x) = \begin{cases} -1 & \text{if } -2 < x \le -1 \\ -2(x+1) & \text{if } -1 < x \le 0 \\ 2(x-1) & \text{if } 0 < x \le 1 \\ -1 & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, we conclude that $f' = g + \delta_{-1} + 2\delta_0 + \delta_1$. Note that this corresponds exactly to the formula found during the lecture.

• For the function j, we have for any $\phi \in \mathcal{D}(\mathbb{R})$,

$$\langle j', \phi \rangle = -\langle j, \phi' \rangle = -\int_{-\infty}^{0} x \phi'(x) dx - \int_{0}^{\infty} (1+x) \phi'(x) dx$$

$$= \int_{-\infty}^{0} \phi(x)dx + \int_{0}^{\infty} \phi(x)dx + \phi(0)$$
$$= \int_{-\infty}^{\infty} \phi(x)dx + \phi(0).$$

We conclude that $j' = 1 + \delta_0$. Note that we can also write j(x) = x + H(x), where H is the heaviside function.

Exercise 2 Let $f : \mathbb{R} \to \mathbb{R}$ be a function with period 1 and

$$f(x) = x, \qquad 0 < x < 1.$$

Find the distributional derivative.

Solution 2 For a test function $\phi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle = -\sum_{n \in \mathbb{Z}} \int_{n}^{n+1} (x - n)\phi'(x) dx,$$

Here, we have used that the function looks like x - n over each interval (n, n - 1): over the interval from n to n + 1, it looks like a linear function with slope 1 that equals 0 at x = n. Now, use integration by parts:

$$-\sum_{n\in\mathbb{Z}} \int_{n}^{n+1} (x-n)\phi'(x)dx = -\sum_{n\in\mathbb{Z}} \left[\phi(n+1) - \int_{n}^{n+1} \phi(x)dx \right]$$
$$= -\sum_{n\in\mathbb{Z}} \phi(n+1) + \sum_{n\in\mathbb{Z}} \int_{n}^{n+1} \phi(x)dx$$
$$= -\sum_{n\in\mathbb{Z}} \phi(n) + \sum_{n\in\mathbb{Z}} \int_{n}^{n+1} \phi(x)dx$$
$$= -\sum_{n\in\mathbb{Z}} \phi(n) + \int_{-\infty}^{\infty} \phi(x)dx.$$

On a technical note, we notice that the sum only involves a finite number of terms and that the integral is finite, too, because ϕ is smooth with compact support. Hence, we have the distributional derivative equals the constant function 1 minus the Dirac comb.

$$f' = 1 - \Delta_1$$
.

We observe that this accounts for the fact that f is a piecewise linear function with slope equal 1 on the intervals and with jumps -1 at each of the interval boundaries.

Exercise 3 Compute the Fourier coefficients of the function f that has period $T = 2\pi$ and satisfies

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 0 & \pi < x \le 2\pi \end{cases}$$

Solution 3 We notice that the period T equals $T = 2\pi$. We have defined the first coefficient to be just the average:

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{2\pi} \int_0^{\pi} dx = \frac{1}{2}.$$

For the other coefficients, we use the formulas from the lecture:

$$a_n = \frac{2}{2\pi} \int_0^{2\pi} f(x) \cos\left(\frac{2\pi nx}{T}\right) dx$$

$$= \frac{2}{2\pi} \int_0^{\pi} \cos\left(\frac{2\pi nx}{T}\right) dx$$

$$= \frac{2}{2\pi} \left[\frac{T}{2\pi n} \sin\left(\frac{2\pi nx}{T}\right)\right]_{x=0}^{x=\pi}$$

$$= \frac{2}{2\pi} \frac{T}{2\pi n} \left(\sin\left(\frac{2\pi n\pi}{T}\right) - \sin(0)\right)$$

$$= \frac{2}{2\pi} \frac{2\pi}{2\pi n} \left(\sin(n\pi) - \sin(0)\right)$$

$$= 0$$

because sin equals zero at integer multiples of π . For the sine mode coefficients, we calculate similarly:

$$b_n = \frac{2}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx$$

$$= \frac{1}{\pi} \left[-\frac{\cos(nx)}{n} \right]_{x=0}^{x=\pi}$$

$$= -\frac{1}{\pi} \frac{1}{n} \left[\cos(n\pi) - \cos(0) \right]$$

$$= -\frac{1}{n\pi} \left[(-1)^n - 1 \right]$$

$$= \frac{1}{n\pi} \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$

Thus we find

$$\frac{a_0}{2} = 0.5, \qquad a_1 = a_2 = \dots = 0,$$
 (1)

and either $b_n = 0$ if n is even and $b_n = \frac{2}{n\pi}$ if n is odd.

Exercise 4 Compute the Fourier coefficients of the following functions, which have period T = 1 and have the given values over the interval [0,1):

$$f(x) = x^2$$

$$g(x) = (1 - x)x$$

$$h(x) = |\sin(2\pi x)|$$

Solution 4 • To compute the Fourier coefficients of $f(x) = x^2$ over the interval [0,1) with a period of 1, use the formulas from the lecture:

$$\frac{a_0}{2} = \int_0^1 f(x)dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

For other coefficients, we apply integration by parts twice, and we use that sin vanishes at multiples of π :

$$\begin{split} a_n &= \frac{2}{T} \int_0^1 x^2 \cos(2\pi n x) dx \\ &= \frac{2}{T} \left[x^2 \frac{\sin(2\pi n x)}{2\pi n} \right]_{x=0}^{x=1} - \frac{2}{T} \int_0^1 \frac{2x \sin(2\pi n x)}{2\pi n} dx \\ &= \frac{2}{T} \left[x^2 \frac{\sin(2\pi n x)}{2\pi n} \right]_{x=0}^{x=1} - \frac{4}{T} \int_0^1 x \frac{\sin(2\pi n x)}{2\pi n} dx \\ &= \frac{2}{T} \left[x^2 \frac{\sin(2\pi n x)}{2\pi n} \right]_{x=0}^{x=1} - \frac{4}{T} \left[x \frac{-\cos(2\pi n x)}{4\pi^2 n^2} \right]_{x=0}^{x=1} + \frac{4}{T} \int_0^1 \frac{-\cos(2\pi n x)}{4\pi^2 n^2} dx \\ &= \frac{2}{T} \left[x^2 \frac{\sin(2\pi n x)}{2\pi n} \right]_{x=0}^{x=1} - \frac{4}{T} \left[x \frac{-\cos(2\pi n x)}{4\pi^2 n^2} \right]_{x=0}^{x=1} + \frac{4}{T} \left[\frac{-\sin(2\pi n x)}{8\pi^3 n^3} \right]_{x=0}^{x=1} \\ &= \frac{1}{\pi^2 n^2} \end{split}$$

$$b_n = \frac{2}{T} \int_0^1 x^2 \sin(2\pi nx) dx$$

$$= \frac{2}{T} \left[-x^2 \frac{\cos(2\pi nx)}{2\pi n} \right]_{x=0}^{x=1} + \frac{2}{T} \int_0^1 \frac{2x \cos(2\pi nx)}{2\pi n} dx$$

$$= \frac{2}{T} \left[-x^2 \frac{\cos(2\pi nx)}{2\pi n} \right]_{x=0}^{x=1} + \frac{4}{T} \int_0^1 \frac{x \cos(2\pi nx)}{2\pi n} dx$$

$$\begin{split} &=\frac{2}{T}\left[-x^2\frac{\cos(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{4}{T}\left[x\frac{\sin(2\pi nx)}{4\pi^2 n^2}\right]_{x=0}^{x=1} - \frac{4}{T}\int_0^1\frac{\sin(2\pi nx)}{4\pi^2 n^2}dx\\ &=\frac{2}{T}\left[-x^2\frac{\cos(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{4}{T}\left[x\frac{\sin(2\pi nx)}{4\pi^2 n^2}\right]_{x=0}^{x=1} + \frac{4}{T}\left[\frac{\cos(2\pi nx)}{8\pi^3 n^3}\right]_{x=0}^{x=1}\\ &=-\frac{2}{T}\frac{1}{2\pi n} + \frac{4}{T}\frac{1}{8\pi^3 n^3} - \frac{4}{T}\frac{1}{8\pi^3 n^3} = -\frac{1}{\pi n} \end{split}$$

• We notice that the function g(x) = (1-x)x can be written g(x) = x - f(x) over the interval [0,1). In other words, g is the difference of the sawtooth function (which equals x over [0,1) and repeats with period T=1), and the periodic function f(x) seen earlier.

We will only need to determine the Fourier coefficients of $\tilde{g}(x) = x$ since we already computed the Fourier coefficients of f(x). We calculate the average

$$\frac{a_0^{\tilde{g}}}{2} = \int_0^1 \tilde{g}(x)dx = \int_0^1 xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

We calculate

$$a_n^{\tilde{g}} = \frac{2}{T} \int_0^1 x \cos(2\pi nx) dx$$

$$= \frac{2}{T} \left[x \frac{\sin(2\pi nx)}{2\pi n} \right]_{x=0}^{x=1} - \frac{2}{T} \int_0^1 \frac{\sin(2\pi nx)}{2\pi n} dx$$

$$= \frac{2}{T} \left[x \frac{\sin(2\pi nx)}{2\pi n} \right]_{x=0}^{x=1} + \frac{2}{T} \left[\frac{\cos(2\pi nx)}{4\pi^2 n^2} \right]_{x=0}^{x=1}$$

$$= \frac{2}{T} \frac{1}{2\pi n} \left(\sin(2\pi n) - \sin(0) \right) + \frac{2}{T} \frac{1}{4\pi^2 n^2} \left(\cos(2\pi n) - \cos(0) \right)$$

$$= 0$$

Lastly,

$$\begin{split} b_n^{\tilde{g}} &= \frac{2}{T} \int_0^1 x \sin(2\pi n x) dx \\ &= \frac{2}{T} \left[-x \frac{\cos(2\pi n x)}{2\pi n} \right]_{x=0}^{x=1} + \frac{2}{T} \int_0^1 \frac{\cos(2\pi n x)}{2\pi n} dx \\ &= \frac{2}{T} \left[-x \frac{\cos(2\pi n x)}{2\pi n} \right]_{x=0}^{x=1} + \frac{2}{T} \left[\frac{\sin(2\pi n x)}{4\pi^2 n^2} \right]_{x=0}^{x=1} \\ &= -\frac{2}{T} \frac{1}{2\pi n} = -\frac{1}{\pi n} \end{split}$$

Now the Fourier coefficients of g(x) are given by

$$b_n = -\frac{1}{\pi n} - -\frac{1}{\pi n} = 0$$

$$a_n = 0 - \frac{1}{\pi^2 n^2} = -\frac{1}{\pi^2 n^2}$$
$$\frac{a_0}{2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

• Note that h(x) is an even function. We first check that the sine modes are have coefficients equal zero.

$$b_n = \frac{2}{T} \int_0^T |\sin(2\pi x)| \sin(2\pi nx) dx$$

$$= 2 \int_0^1 |\sin(2\pi x)| \sin(2\pi nx) dx$$

$$= 2 \int_0^{\frac{1}{2}} \sin(2\pi x) \sin(2\pi nx) dx - 2 \int_{\frac{1}{2}}^1 \sin(2\pi x) \sin(2\pi nx) dx.$$

One possibility to proceed is via substitution in the second integral, setting $x = y + \frac{1}{2}$. We then find

$$\int_{\frac{1}{2}}^{1} \sin(2\pi x) \sin(2\pi nx) dx$$

$$= \int_{0}^{\frac{1}{2}} \sin(2\pi (x+0.5)) \sin(2\pi n(x+0.5)) dx$$

$$= \int_{0}^{\frac{1}{2}} \sin(2\pi x + \pi) \sin(2\pi nx + \pi n) dx = (-1)^{n+1} \int_{0}^{\frac{1}{2}} \sin(2\pi x) \sin(2\pi nx + \pi n) dx.$$

In the last step, we have used the symmetry properties of the sine function. Hence

$$b_n = 2\left(1 - (-1)^{n+1}\right) \int_0^{\frac{1}{2}} \sin(2\pi x) \sin(2\pi nx) dx.$$

If n is odd, then the last expression is already zero. Otherwise, if n is even, we want to show that last integral is zero. We use the sum formula:

$$\sin(2\pi x)\sin(2\pi nx) = \frac{1}{2}\cos(2\pi(n-1)x) - \frac{1}{2}\cos(2\pi(n+1)x).$$

With that observation, we find

$$b_n = \frac{1}{2} \int_0^{\frac{1}{2}} \cos(2\pi(n-1)x) - \cos(2\pi(n+1)x) dx$$
$$= \frac{1}{2} \left[\frac{\sin(2\pi(n-1)x)}{2\pi(n-1)} - \frac{\sin(2\pi(n+1)x)}{2\pi(n+1)} \right]_{x=0}^{x=\frac{1}{2}}.$$

For x = 0, the two sine terms are already zero, and for $x = \frac{1}{2}$, the two sine terms are evaluated at integer multiples of π and hence equal zero, too. We conclude that $b_n = 0$. Next, we calculate the average

$$\frac{a_0}{2} = \frac{1}{T} \int_0^1 |\sin 2\pi x| dx = \frac{2}{T} \int_0^{\frac{1}{2}} \sin 2\pi x dx = \frac{2}{T} \left[\frac{-1}{2\pi} \cos 2\pi x \right]_0^{\frac{1}{2}} = \frac{2}{\pi}.$$

The coefficients a_n for the cosine modes with $n \geq 1$ are:

$$a_n = \frac{4}{T} \int_0^{\frac{1}{2}} \sin 2\pi x \cos(2\pi nx) dx$$

$$= \frac{4}{T} \int_0^{\frac{1}{2}} \frac{1}{2} \sin 2\pi (1 - n)x + \frac{1}{2} \sin 2\pi (n + 1)x dx$$

$$= \frac{4}{T} \int_0^{\frac{1}{2}} -\frac{1}{2} \sin 2\pi (n - 1)x + \frac{1}{2} \sin 2\pi (n + 1)x dx$$

$$= \left[\frac{\cos 2\pi (n - 1)x}{\pi (n - 1)} \right]_{x=0}^{x=\frac{1}{2}} + \left[-\frac{\cos 2\pi (n + 1)x}{\pi (n + 1)} \right]_{x=0}^{x=\frac{1}{2}}$$

$$= \frac{(-1)^{n-1}}{\pi (n - 1)} - \frac{1}{\pi (n - 1)} - \frac{(-1)^{n+1}}{\pi (n + 1)} + \frac{1}{\pi (n + 1)}$$

$$= -\frac{1 + (-1)^n}{\pi (n - 1)} + \frac{1 - (-1)^{n+1}}{\pi (n + 1)}$$

$$= -\frac{1 + (-1)^n}{\pi (n - 1)} + \frac{1 + (-1)^n}{\pi (n + 1)}$$

Therefore, when $n \geq 1$:

$$a_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ -\frac{2}{\pi(n-1)} + \frac{2}{\pi(n+1)}, & \text{if } n \text{ is even,} \end{cases}$$