Analysis III - 203(d)

Winter Semester 2024

Session 13: December 12, 2024

Exercise 1 The Fourier transform of

$$f(x) = e^{-5x^2}$$

is the function

$$\hat{f}(\alpha) = \frac{1}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}.$$

Find the Fourier transforms of

$$f'$$
, f'' , f''' , f'''' , $g(x) = f(2x)$, $h(x) = f(x-3)$.

Solution 1 To evaluate the Fourier transform of the derivative of a function we will use the following identity from the lecture:

$$\mathfrak{F}\left(f^{(n)}\right)(\alpha) = (i\alpha)^n \mathfrak{F}(f)(\alpha)$$

We then find

•
$$\mathfrak{F}(f')(\alpha) = i\alpha\mathfrak{F}(f)(\alpha) = \frac{i\alpha}{\sqrt{10}}e^{-\frac{\alpha^2}{20}}$$

•
$$\mathfrak{F}(f'')(\alpha) = -\alpha^2 \mathfrak{F}(f)(\alpha) = -\frac{\alpha^2}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}$$

•
$$\mathfrak{F}(f''')(\alpha) = -i\alpha^3 \mathfrak{F}(f)(\alpha) = \frac{-i\alpha^3}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}$$

•
$$\mathfrak{F}(f'''')(\alpha) = \alpha^4 \mathfrak{F}(f)(\alpha) = \frac{\alpha^4}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}$$

For the Fourier transform of g(x) we use the following identity from the lecture: $g(x) = e^{-ibx} f(ax)$ has the Fourier transform $\hat{g}(\alpha) = \frac{1}{|a|} \hat{f}\left(\frac{\alpha+b}{a}\right)$, consequently:

$$\mathfrak{F}(g(x))(\alpha) = \mathfrak{F}(f(2x))(\alpha)$$

$$= \frac{1}{2}\mathfrak{F}(f(x))\left(\frac{\alpha}{2}\right)$$

$$= \frac{1}{2\sqrt{10}}e^{-\frac{\left(\frac{\alpha}{2}\right)^2}{20}}$$

$$=\frac{1}{2\sqrt{10}}e^{-\frac{\alpha^2}{80}}$$

For the Fourier transform of h(x) we use a change of variables:

$$\mathfrak{F}(h(x))(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-5(x-3)^2} e^{-ix\alpha} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-5x^2} e^{-i(x+3)\alpha} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-5x^2} e^{-ix\alpha} e^{-i3\alpha} \, dx$$

$$= \frac{e^{-i3\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-5x^2} e^{-ix\alpha} \, dx$$

$$= e^{-i3\alpha} \mathfrak{F}(f(x))(\alpha)$$

$$= \frac{e^{-i3\alpha}}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}.$$

$$= \frac{1}{\sqrt{10}} e^{-\frac{\alpha^2}{20} - i3\alpha}.$$

Exercise 2 Find $f: \mathbb{R} \to \mathbb{R}$ such that

$$\hat{f}(\alpha) = \frac{3}{1+\alpha^2} + \frac{-1}{1+4\alpha^2} + \frac{\sin(4\alpha+3)}{4\alpha+3}$$

Solution 2 Because of the linearity property we can treat the inverse of each term in separately. Therefore First,

$$\mathfrak{F}^{-1}\left(\frac{\sin(4\alpha+3)}{4ar+3}\right)(x) = \frac{\frac{1}{4}}{\frac{1}{4}}\mathfrak{F}^{-1}\left(\frac{\sin\left(\frac{\alpha+3/4}{\frac{1}{4}}\right)}{\frac{\alpha+\frac{3}{4}}{\frac{1}{4}}}\right)(x)$$

$$= \frac{1}{4}e^{-i\frac{3}{4}x}\sqrt{\frac{\pi}{2}}\mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{\sin(\alpha)}{\alpha}\right)\left(\frac{x}{4}\right)$$

$$= \begin{cases} \frac{1}{4}\sqrt{\frac{\pi}{2}}e^{-i\frac{3}{4}x} & if -4 < x < 4\\ 0 & else \end{cases}$$

Second,

$$\mathfrak{F}^{-1}\left(\frac{3}{1+\alpha^2}\right)(x) = 3\sqrt{\frac{\pi}{2}}\mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+a^2}\right)(x)$$
$$= 3\sqrt{\frac{\pi}{2}}e^{-|x|}$$

Lastly,

$$\mathfrak{F}^{-1}\left(\frac{-1}{1+4\alpha^2}\right)(x) = -\sqrt{\frac{\pi}{2}}\mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+(2\alpha)^2}\right)(x)$$

$$= -\sqrt{\frac{\pi}{2}}\frac{\frac{1}{2}}{\frac{1}{2}}\mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+\left(\frac{\alpha}{\frac{1}{2}}\right)^2}\right)(x \mid x)$$

$$= -\sqrt{\frac{\pi}{2}}\frac{1}{2}\mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+\alpha^2}\right)\left(\frac{x}{2}\right)$$

$$= -\frac{1}{2}\sqrt{\frac{\pi}{2}}e^{-\left|\frac{x}{2}\right|}.$$

Altogether this gives

$$f(x) = \begin{cases} -\frac{1}{2}\sqrt{\frac{\pi}{2}}e^{-\left|\frac{x}{2}\right|} + 3\sqrt{\frac{\pi}{2}}e^{-\left|x\right|} + \frac{1}{4}\sqrt{\frac{\pi}{2}}e^{-i\frac{3}{4}x} & -4 \le x \le 4\\ -\frac{1}{2}\sqrt{\frac{\pi}{2}}e^{-\left|\frac{x}{2}\right|} + 3\sqrt{\frac{\pi}{2}}e^{-\left|x\right|} & else \end{cases}$$

Exercise 3 We consider the Poisson problem with Dirichlet boundary conditions over the interval [0, L]:

$$-\Delta u(x) = x^2, \quad 0 < x < L,$$

 $u(0) = 1, \quad u(L) = 2$

- Solve this problem directly. The solution is a polynomial of order 4.
- Extend the right-hand side $f(x) = x^2$ to an odd function with period 2L and compute its Fourier coefficients.
- Using these coefficients, find the Fourier series of the solution u^f , which solves

$$-\Delta u^f(x) = x^2, \quad 0 < x < L,$$

 $u^f(0) = 0, \quad u^f(L) = 0.$

How do you use the superposition principle to solve the full problem?

Solution 3 • According to the hint, u(x) must be of the form $u(x) = ax^4 + bx^3 + cx^2 + dx + e$ for some coefficients $a, b, c, d, e \in \mathbb{R}$. Inserting this ansatz into the Poisson equation and the boundary conditions, we find the system of equations

$$u(0) = e = 1, (1)$$

$$u(L) = aL^4 + bL^3 + cL^2 + dL + e = 2,$$
(2)

$$-\Delta u(x) = -12ax^2 - 6bx - 2c = x^2. (3)$$

By matching the coefficients of the polynomials on the right- and left-hand side in (3), we find

$$-12a = 1,$$

 $-6b = 0,$
 $-2c = 0.$

Hence, we have $a = -\frac{1}{12}, b = 0, c = 0$. Therefore, (2) simplifies to

$$-\frac{1}{12}L^4 + dL + 1 = 2,$$

from which we deduce that $d = \frac{1}{12}L^3 + \frac{1}{L}$. Thus, the solution to the Poisson problem is

$$u(x) = -\frac{1}{12}x^4 + \left(\frac{1}{12}L^3 + \frac{1}{L}\right)x + 1.$$

• We define the odd extension of $f(x) = x^2$ as the function \tilde{f} with period 2L that satisfies

$$\tilde{f}(x) = \begin{cases} x^2 & \text{if } 0 \le x \le L, \\ -x^2 & \text{if } -L \le x < 0. \end{cases}$$

By construction, \tilde{f} is an odd function with period 2L. Denote by \tilde{a}_n and \tilde{b}_n the Fourier coefficients of \tilde{f} . We immediately have $\tilde{a}_n = 0$ for all $n \in \mathbb{N}$ because the function is odd. Moreover, by some simple integration by parts

$$\tilde{b}_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \sin\left(\frac{\pi n}{L}x\right) dx = \frac{2}{L} \int_{0}^{L} x^2 \sin\left(\frac{\pi n}{L}x\right) dx$$
$$= -\frac{2L^2(2 + (-1)^n((n\pi)^2 - 2))}{(n\pi)^3}.$$

We conclude that

$$\tilde{f}(x) = \sum_{n \ge 1} \tilde{b}_n \sin\left(\frac{\pi n}{L}x\right) = \sum_{n \ge 1} -\frac{2L^2(2 + (-1)^n((n\pi)^2 - 2))}{(n\pi)^3} \sin\left(\frac{\pi n}{L}x\right).$$

• To apply the superposition principle, we split the problem into two Poisson problems: on the one hand.

$$-\Delta u^f(x) = x^2, \quad 0 < x < L,$$

 $u^f(0) = 0, \quad u^f(L) = 0,$

and on the other hand,

$$-\Delta u^g(x) = 0, \quad 0 < x < L,$$

 $u^g(0) = 1, \quad u^g(L) = 2.$

The full solution will be the sum of u^f and u^g .

We define

$$u^{f}(x) = -\sum_{n=1}^{\infty} \frac{\tilde{b}_{n}}{-\pi^{2} n^{2} / L^{2}} \sin(\pi n x / L)$$
(4)

$$= \sum_{n=1}^{\infty} \frac{\tilde{b}_n}{\pi^2 n^2 / L^2} \sin(\pi n x / L).$$
 (5)

Differentation, distributed over the Fourier modes, now shows that

$$-\partial_{xx}u^f(x) = f(x). (6)$$

Additionally, u^f satisfies the homogeneous Dirichlet boundary conditions.

We can find u^g in a manner similar to the previous step. Since its second-derivative is zero, u^g must be a linear function over [0, L], having the form:

$$u^g(x) = dx + e. (7)$$

The boundary conditions now lead to $u^g(x) = \frac{1}{L}x + 1$

By the superposition principle, the solution to the original problem is

$$u(x) = u^f(x) + u^g(x). (8)$$

Exercise 4 Directly compute the solution of the problem

$$-\Delta u(x) = x^2, \quad 0 < x < L,$$

 $u(0) = 0, \quad u(L) = 0,$

using elementary analysis. Then find the Fourier coefficients of its odd extension to the interval [-L, L]. Compare this with the function u^f from the previous exercise.

Solution 4 As in the previous exercise, the solution must be of the form $u(x) = ax^4 + bx^3 + cx^2 + dx + e$ for some coefficients $a, b, c, d, e \in \mathbb{R}$. Inserting this ansatz into the Poisson equation and the boundary conditions, we find the system of equations

$$u(0) = e = 0, (9)$$

$$u(L) = aL^4 + bL^3 + cL^2 + dL + e = 0, (10)$$

$$-\Delta u(x) = -12ax^2 - 6bx - 2c = x^2. \tag{11}$$

By matching the coefficients of the polynomials on the right- and left-hand side in (11), we find

$$-12a = 1$$
, $-6b = 0$, $-2c = 0$.

Hence, we have $a = -\frac{1}{12}$, b = 0, and c = 0. Therefore, (10) simplifies to

$$-\frac{1}{12}L^4 + dL = 0,$$

from which we deduce that $d=\frac{1}{12}L^3$. Thus, the solution to the Poisson problem is

$$u(x) = -\frac{1}{12}x^4 + \frac{1}{12}L^3x.$$

To check that this solution is equal to the one found in the previous exercise, we compute the Fourier coefficients of the odd extension \tilde{u} of u to the interval [-L, L] defined by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } 0 \le x \le L, \\ -u(-x) & \text{if } -L \le x < 0. \end{cases}$$

By definition, \tilde{u} is an odd function, and therefore the Fourier series takes the form

$$\tilde{u}(x) = \sum_{n \ge 1} \tilde{u}_n \sin\left(\frac{\pi n}{L}x\right),$$

where \tilde{u}_n is the Fourier coefficient of the n-th cosine mode. We compute

$$\tilde{u}_n = \frac{2}{L} \int_0^L u(x) \sin\left(\frac{\pi n}{L}x\right) dx$$

$$= \frac{2}{L} \int_0^L \left(-\frac{1}{12}x^4 + \frac{1}{12}L^3x\right) \sin\left(\frac{\pi n}{L}x\right) dx$$

$$= -\frac{2L^4(2 + (-1)^n)((n\pi)^2 - 2)}{(n\pi)^5}.$$

Note that we have $\tilde{u}_n = \left(\frac{L}{\pi n}\right)^2 \tilde{b}_n$, where \tilde{b}_n is as in the previous exercise.

Exercise 5 We solve the Poisson problem with homogeneous Dirichlet boundary conditions over [0, 1]:

$$-\Delta u(x) = \begin{cases} x & \text{if } 0 < x \le 0.5 \\ 0 & \text{if } 0.5 < x \le 1 \end{cases}, \qquad 0 < x < 1,$$
$$u(0) = 0, \quad u(1) = 0$$

- Extend the right-hand side f(x) to an odd function with period 2 and compute its Fourier coefficients.
- Using these coefficients, find the solution u. Verify that the boundary condition u(0) = u(1) = 0 is satisfied.

Solution 5 • We define the odd extension of f(x) as

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 \le x < 1, \\ -f(-x) & \text{if } -1 < x < 0. \end{cases}$$

As in the previous exercises, we find that

$$\tilde{f}(x) = \sum_{n>1} \tilde{f}_n \sin(\pi n x),$$

using that \tilde{f} is odd. The Fourier coefficients of the sine modes are

$$\tilde{f}_n = 2\int_0^1 f(x)\sin(\pi nx)dx = 2\int_0^{\frac{1}{2}} x\sin(\pi nx)dx = \frac{-\pi n\cos(\frac{\pi n}{2}) + 2\sin(\frac{\pi n}{2})}{(\pi n)^2}.$$

One can check that

$$\tilde{f}_n = \begin{cases} \frac{2}{(\pi n)^2} (-1)^n, & \text{if } n \text{ is odd,} \\ -\frac{1}{\pi n} (-1)^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

• We define the solution in terms of its Fourier series:

$$u(x) = -\sum_{n=1}^{\infty} \frac{\tilde{f}_n}{-\pi^2 n^2} \sin(\pi n x) = \sum_{n=1}^{\infty} \frac{\tilde{f}_n}{\pi^2 n^2} \sin(\pi n x).$$
 (12)

Clearly, u satisfies the homogeneous Dirichlet boundary conditions, and differentiation provides the desired differential equation

$$-u''(x) = f(x). (13)$$

This shows that u solves the Poisson problem. Note that u is not in the form of a Fourier series over the interval [0, L] but in the form of the Fourier series of its odd extension over [-L, L]. The boundary conditions are satisfied because u is a Fourier sine series.

Exercise 6 Consider the following Fourier sine series for two functions $u, f : \mathbb{R} \to \mathbb{R}$ with period T:

$$u(x) = \sum_{n=1}^{\infty} b_n^u \sin\left(\frac{2\pi n}{T}x\right), \quad f(x) = \sum_{n=1}^{\infty} b_n^f \sin\left(\frac{2\pi n}{T}x\right).$$

Express the Fourier coefficients of f by the Fourier coefficients of u, and vice-versa, express the Fourier coefficients of u by the Fourier coefficients of f, if u and f are related by the following differential equations:

$$(a) -u''(x) = f(x).$$

(b)
$$u''''(x) = f(x)$$
.

(c)
$$-u''(x) + 25u(x) = f(x)$$
.

(d)
$$u''''(x) - u''(x) = f(x)$$
.

(e)
$$-u''(x) + \gamma \cdot u''(x) = f(x)$$
.

In the last item, $\gamma \in \mathbb{R}$ is a constant. For which values of γ can you always find a solution?

Solution 6 • We have

$$-u''(x) = -\sum_{n=1}^{\infty} b_n^u \left(\frac{2\pi n}{T}\right)^2 \sin\left(\frac{2\pi n}{T}x\right).$$

Matching coefficients leads to the relationships

$$b_n^u \left(\frac{2\pi n}{T}\right)^2 = b_n^f, \quad b_n^u = \left(\frac{2\pi n}{T}\right)^{-2} b_n^f.$$

• We have

$$u''''(x) = \sum_{n=1}^{\infty} b_n^u \left(\frac{2\pi n}{T}\right)^4 \sin\left(\frac{2\pi n}{T}x\right).$$

Matching coefficients leads to the relationships

$$b_n^u \left(\frac{2\pi n}{T}\right)^4 = b_n^f, \quad b_n^u = \left(\frac{2\pi n}{T}\right)^{-4} b_n^f.$$

$$-u''(x) + 25u = \sum_{n=1}^{\infty} b_n^u \left(-\left(\frac{2\pi n}{T}\right)^2 + 25\right) \sin\left(\frac{2\pi n}{T}x\right).$$

Matching coefficients leads to the relationships

$$b_n^u \left(-\left(\frac{2\pi n}{T}\right)^2 + 25\right) = b_n^f, \quad b_n^u = \left(-\left(\frac{2\pi n}{T}\right)^2 + 25\right)^{-1} b_n^f.$$

$$u''''(x) - u''(x) = \sum_{n=1}^{\infty} b_n^u \left(\left(\frac{2\pi n}{T} \right)^4 - \left(\frac{2\pi n}{T} \right)^2 \right) \sin\left(\frac{2\pi n}{T} x \right).$$

Matching coefficients leads to the relationships

$$b_n^u \left(\left(\frac{2\pi n}{T} \right)^4 - \left(\frac{2\pi n}{T} \right)^2 \right) = b_n^f, \quad b_n^u = \left(\left(\frac{2\pi n}{T} \right)^4 - \left(\frac{2\pi n}{T} \right)^2 \right)^{-1} b_n^f.$$

$$u''''(x) - \gamma \cdot u''(x) = \sum_{n=1}^{\infty} b_n^u \left(\left(\frac{2\pi n}{T} \right)^4 - \gamma \cdot \left(\frac{2\pi n}{T} \right)^2 \right) \sin\left(\frac{2\pi n}{T} x \right).$$

Matching coefficients leads to the relationships

$$b_n^u \left(\left(\frac{2\pi n}{T} \right)^4 - \gamma \cdot \left(\frac{2\pi n}{T} \right)^2 \right) = b_n^f, \quad b_n^u = \left(\left(\frac{2\pi n}{T} \right)^4 - \gamma \cdot \left(\frac{2\pi n}{T} \right)^2 \right)^{-1} b_n^f.$$

Remark: the second-order equation -u'' = f, the Poisson problem, appears in many applications. The fourth-order equation u'''' = f appears in the mathematical theory of elasticity, for example, when modeling the bending of a one-dimensional stick under external force.

Exercise 7 Recall the definition of the Fourier transform:

$$\mathfrak{F}[f](\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-it\alpha} dt \tag{14}$$

Let $a, b, c \in \mathbb{R}$ be real parameters with $a \neq 0$. Compute the Fourier transforms of the following:

$$g(t) = f(at), (15)$$

$$h(t) = e^{-itb} f(t), (16)$$

$$m(t) = f(t - c). (17)$$

Hint: you have the first two in the lecture and in textbook. You can compute them using results from the lecture or via some standard integral manipulations.

Solution 7 We write $\hat{f}(\alpha) = \mathfrak{F}[f](\alpha)$ for the Fourier transform of α . To compute the Fourier transforms of the given functions g(t), h(t), and m(t), we proceed as follows:

• Let $\hat{g}(\alpha) = \mathfrak{F}[g](\alpha)$. By definition,

$$\hat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t)e^{-it\alpha} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(at)e^{-it\alpha} dt.$$

We perform a substitution u = at, so du = |a| dt and $t = \frac{u}{a}$. This gives

$$\hat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)e^{-i\frac{u}{a}\alpha} \frac{du}{a} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a} \int_{-\infty}^{+\infty} f(u)e^{-i\frac{\alpha}{a}u} du.$$

Recognizing the integral as the Fourier transform of f(t), we find:

$$\hat{g}(\alpha) = \frac{1}{|a|} \hat{f}\left(\frac{\alpha}{a}\right),\,$$

• Let $\hat{h}(\alpha) = \mathfrak{F}[h](\alpha)$. Using definitions.

$$\hat{h}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(t)e^{-it\alpha} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ibt} f(t)e^{-it\alpha} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-it(\alpha+b)} dt.$$

We recognizing the integral as the Fourier transform of f(t). Thus,

$$\hat{h}(\alpha) = \hat{f}(\alpha + b).$$

• Let $\hat{m}(\alpha) = \mathfrak{F}[m](\alpha)$. By definition,

$$\hat{m}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} m(t)e^{-it\alpha} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t-c)e^{-it\alpha} dt.$$

We substitute u = t - c, so du = dt and t = u + c. This gives

$$\hat{m}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)e^{-i(u+c)\alpha} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)e^{-iu\alpha}e^{-ic\alpha} du.$$

We pull out the factor out $e^{-ic\alpha}$,

$$\hat{m}(\alpha) = e^{-ic\alpha} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)e^{-iu\alpha} du,$$

and recognize the last integral as the Fourier transform of f(t). In summary,

$$\hat{m}(\alpha) = e^{-ic\alpha} \hat{f}(\alpha).$$