Analysis III - 203(d)

Winter Semester 2024

Session 12: December 5, 2024

Exercise 1 *Consider the following functions with period T:*

- *A function f with period $T = 1$ such that*

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 0.5 \\ 0 & \text{if } 0.5 \leq x \leq 1 \end{cases}$$

- *A function g with period $T = 2\pi$ such that*

$$g(x) = \begin{cases} x & \text{if } 0 \leq x < \pi \\ 2\pi - x & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

- *A function h with period $T = 1$ such that*

$$h(x) = -x \text{ if } 0 \leq x < 1.$$

Find the Fourier coefficients of the Fourier series of these functions. You can use the Fourier series seen in the lecture to get the coefficients.

Solution 1

- *As seen in the lecture we know the Fourier coefficients of a square wave,*

$$l(x) = \begin{cases} 1, & \text{if } 0 \leq x < 0.5 \\ -1, & \text{if } 0.5 \leq x < 1 \end{cases},$$

are given by:

$$a_n = 0 \quad \text{for } n \geq 0, \quad b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

We can express the function $f(x)$ in terms of $l(x)$ as follows:

$$f(x) = \frac{1}{2} + \frac{1}{2} l(x)$$

therefore the Fourier coefficients of f are given by:

$$a_n = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0 \end{cases}, \quad b_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{2}{n\pi}, & \text{if } n \text{ is even} \end{cases}$$

- *As seen in the lecture we know the Fourier coefficients of a triangle wave,*

$$m(x) = \begin{cases} 2x, & \textit{if } 0 \leq x < 0.5 \\ 2(1-x), & \textit{if } 0.5 \leq x < 1 \end{cases}$$

are given by:

$$a_0 = 1, \quad a_n = \begin{cases} -\frac{4}{n^2\pi^2}, & \textit{if } n \textit{ is odd} \\ 0, & \textit{if } n \textit{ is even} \end{cases}, \quad b_n = 0 \textit{ for } n > 0$$

We can express the function $g(x)$ in terms of $m(x)$ as follows:

$$g(x) = \pi m(\frac{x}{2\pi})$$

therefore the Fourier coefficients of g are given by:

$$a_0 = \pi, \quad a_n = \begin{cases} -\frac{4}{n^2\pi}, & \textit{if } n \textit{ is odd} \\ 0, & \textit{if } n \textit{ is even} \end{cases}, \quad b_n = 0 \textit{ for } n > 0$$

- *As seen in the lecture we know the Fourier coefficients of a sawtooth wave,*

$$n(x) = x \textit{ for } 0 \leq x < 1$$

are given by:

$$a_0 = 1, \quad a_n = 0 \textit{ for } n > 0, \quad a_n = -\frac{1}{n\pi} \textit{ for } n > 0,$$

We can express the function $h(x)$ in terms of $n(x)$ as follows:

$$h(x) = -n(x)$$

therefore the Fourier coefficients of h are given by:

$$a_0 = -1, \quad a_n = 0 \textit{ for } n > 0, \quad a_n = \frac{1}{n\pi} \textit{ for } n > 0,$$

Exercise 2 *Explicitly write down the coefficients a_n and b_n and the periods of the following Fourier series:*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \sin(2\pi(2n+1)x),$$

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos(2\pi(2n-1)x),$$

$$h(x) = \frac{\pi}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(\pi n x).$$

Determine the Fourier coefficients of the derivatives of those functions.

Solution 2

- *We substitute $m = 2n+1$ to obtain*

$$f(x) = \sum_{\substack{m=3 \\ m \text{ is odd}}}^{\infty} \frac{1}{m^2} \sin(2\pi m x),$$

therefore the Fourier coefficients are given by

$$a_m = 0 \text{ for } m \geq 0, \quad b_m = \begin{cases} \frac{1}{m^2}, & \text{if } m \text{ is odd and } m \geq 3 \\ 0, & \text{otherwise} \end{cases}$$

and the period is $T = 1$. The derivative of $f(x)$ is given by:

$$f'(x) = \sum_{\substack{m=3 \\ m \text{ is odd}}}^{\infty} \frac{2\pi}{m} \sin(2\pi m x),$$

therefore the Fourier coefficients of the derivative of $f(x)$ are given by

$$b_m = 0 \text{ for } m \geq 0, \quad a_m = \begin{cases} \frac{2\pi}{m}, & \text{if } m \text{ is odd and } m \geq 3 \\ 0, & \text{otherwise} \end{cases}$$

- *We substitute $m = 2n-1$ to obtain*

$$g(x) = \sum_{\substack{m=1 \\ m \text{ is odd}}}^{\infty} \frac{(-1)^{\frac{m+1}{2}}}{m^3} \cos(2\pi m x),$$

therefore the Fourier coefficients are given by

$$b_m = 0 \text{ for } m > 0, \quad a_m = \begin{cases} \frac{(-1)^{\frac{m+1}{2}}}{m^3}, & \text{if } m \text{ is odd and } m \geq 1 \\ 0, & \text{if } m \text{ is even and } m \geq 1 \\ 0 & \text{if } m = 0 \end{cases}$$

and the period is $T = 1$. The derivative of $g(x)$ is given by:

$$g'(x) = \sum_{\substack{m=1 \\ m \text{ is odd}}}^{\infty} -\frac{2\pi(-1)^{\frac{m+1}{2}}}{m^2} \sin(2\pi m x),$$

therefore the Fourier coefficients of the derivative of $g(x)$ are given by

$$a_m = 0 \text{ for } m \geq 0, \quad b_m = \begin{cases} -\frac{2\pi(-1)^{\frac{m+1}{2}}}{m^2}, & \text{if } m \text{ is odd and } m \geq 1 \\ 0, & \text{if } m \text{ is even and } m \geq 1 \\ 0 & \text{if } m = 0 \end{cases}$$

- *the Fourier coefficients are given by*

$$b_n = 0 \text{ for } n > 0, \quad a_n = \begin{cases} \frac{2\pi}{3}, & \text{if } n = 0 \\ -\frac{(-1)^n}{n^2}, & \text{if } n > 1 \end{cases}$$

and the period $T = 2$. The derivative of $h(x)$ is given by:

$$h'(x) = \sum_{n=1}^{\infty} -\frac{\pi(-1)^{n+1}}{n} \sin(\pi n x),$$

therefore the Fourier coefficients of the derivative of $h(x)$ are given by

$$a_n = 0 \text{ for } n \geq 0, \quad b_n = -\frac{\pi(-1)^{n+1}}{n}, \quad \text{for } n > 0$$

Exercise 3 *Let $f(x)$ be a periodic function with period T, represented by its Fourier series:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right) \right).$$

As explained in the lecture,

$$\int_{x_0}^{x} f(x)dx = \frac{a_0}{2}(x - x_0) + \sum_{n=1}^{\infty} a_n \int_{x_0}^{x} \cos\left(\frac{2\pi n x}{T}\right) dx + b_n \int_{x_0}^{x} \sin\left(\frac{2\pi n x}{T}\right) dx.$$

Complete this discussion and find the Fourier series of a function $g(x)$ such that

$$\int_{x_0}^{x} f(x) = cx + g(x)$$

for some $c \in \mathbb{R}$.

Solution 3 *We first compute*

$$\int_{x_0}^{x} \frac{a_0}{2}\, dt = \frac{a_0}{2}(x - x_0).$$

We integrate the sign and cosine terms, using the fundamental theorem of calculus:

$$\int_{x_0}^{x} \cos\left(\frac{2\pi n t}{T}\right) dt = \frac{T}{2\pi n} \left(\sin\left(\frac{2\pi n x}{T}\right) - \sin\left(\frac{2\pi n x_0}{T}\right) \right).$$

$$\int_{x_0}^{x} \sin\left(\frac{2\pi n t}{T}\right) dt = -\frac{T}{2\pi n} \left(\cos\left(\frac{2\pi n x}{T}\right) - \cos\left(\frac{2\pi n x_0}{T}\right) \right).$$

Combining all terms, the integral $F(x)$ becomes:

$$F(x) = \frac{a_0}{2}(x - x_0) + \sum_{n=1}^{\infty} a_n \frac{T}{2\pi n}\left(\sin\left(\frac{2\pi n x}{T}\right) - \sin\left(\frac{2\pi n x_0}{T}\right)\right) - b_n \frac{T}{2\pi n}\left(\cos\left(\frac{2\pi n x}{T}\right) - \cos\left(\frac{2\pi n x_0}{T}\right)\right).$$

We rewrite this once again and obtain

$$\begin{aligned} F(x) = \frac{a_0}{2}x - \frac{a_0 x_0}{2} &+ \sum_{n=1}^{\infty}\left((-a_n)\frac{T}{2\pi n}\sin\left(\frac{2\pi n x_0}{T}\right) + b_n \frac{T}{2\pi n}\cos\left(\frac{2\pi n x_0}{T}\right)\right) \\ &+ \sum_{n=1}^{\infty}(-b_n)\frac{T}{2\pi n}\cos\left(\frac{2\pi n x}{T}\right) + a_n \frac{T}{2\pi n}\sin\left(\frac{2\pi n x}{T}\right). \end{aligned}$$

We thus obtain $c = \frac{a_0}{2}$ and the Fourier coefficients of the function g:

$$\begin{aligned} \frac{A_0}{2} &:= -\frac{a_0 x_0}{2} + \sum_{n=1}^{\infty}\left((-a_n)\frac{T}{2\pi n}\sin\left(\frac{2\pi n x_0}{T}\right) + b_n \frac{T}{2\pi n}\cos\left(\frac{2\pi n x_0}{T}\right)\right) \\ A_n &:= (-b_n)\frac{T}{2\pi n} \\ B_n &:= a_n \frac{T}{2\pi n} \end{aligned}$$

This completes the discussion.

Exercise 4 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions. Recall that a function is called even if*

$$f(-x) = f(x)$$

and odd if

$$f(-x) = -f(x)$$

- *Show that if f and g are both odd or both even, then fg is even*
- *Show that if one of f and g is odd and the other is even, then fg is odd.*
- *Show that the only function that is both odd and even has constant value zero.*

Solution 4

- *If f and g are both even, then*

  $$f(-x)g(-x) = f(x)g(x),$$

  and if both are odd, then

  $$f(-x)g(-x) = (-1)^2 f(x)g(x) = f(x)g(x),$$

  so fg must be an even function.

- *Suppose that f is even and that g is odd. Then*

$$f(-x)g(-x) = f(x)g(-x) = -f(x)g(x),$$

showing that fg is an odd function. The same is true if we switch the role of f and g.

- *Suppose that f is both odd and even. Then*

$$f(x) = f(-x) = -f(x)$$

for each $x \in \mathbb{R}$. But the only number that equals its negative is zero. So $f(x) = 0$ for each x.

Exercise 5 *Compute the Fourier transform of the function*

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

You can either directly use the complex exponential, or you can express it in terms of the sine and cosine function.
(Interpretation: the function $f(x)$ describes a localized signal: it is zero at $x = 0$, then it rises linearly up to 1, and then it jumps back to zero and remains zero from there on. The signal is not periodic.)

Solution 5 *We write down the solution in two different ways, either using the complex exponential directly, or writing it as a sum of sine and cosine.*

$$\begin{aligned}
\mathfrak{F}(f)(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^1 x e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{x}{-i\alpha} e^{-i\alpha x} \right]_0^1 - \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{1}{-i\alpha} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{x}{-i\alpha} e^{-i\alpha x} \right]_0^1 + \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{1}{i\alpha} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{x}{-i\alpha} e^{-i\alpha x} \right]_0^1 + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha^2} e^{-i\alpha x} \right]_0^1 \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{-i\alpha} e^{-i\alpha} + \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\alpha^2} e^{-i\alpha} - \frac{1}{\alpha^2} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{i}{\alpha} e^{-i\alpha} + \frac{1}{\alpha^2} e^{-i\alpha} - \frac{1}{\alpha^2} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{i}{\alpha} \cos\alpha + \frac{1}{\alpha} \sin\alpha + \frac{1}{\alpha^2} \cos\alpha - \frac{i}{\alpha^2} \sin\alpha - \frac{1}{\alpha^2} \right)
\end{aligned}$$

Next we do it in terms of sine and cosine functions.

$$\begin{aligned}
\mathfrak{F}(f)(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(\cos \alpha x - i \sin \alpha x) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \alpha x dx - i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \sin \alpha x dx
\end{aligned}$$

We evaluate each integral seperately.

$$\begin{aligned}
\int_0^1 x \cos(\alpha x) dx &= \left[\frac{x}{\alpha} \sin \alpha x\right]_0^1 - \int_0^1 \frac{1}{\alpha} \sin \alpha x dx \\
&= \left[\frac{x}{\alpha} \sin \alpha x\right]_0^1 + \left[\frac{1}{\alpha^2} \cos \alpha x\right]_0^1 \\
&= \frac{1}{\alpha} \sin \alpha + \frac{1}{\alpha^2} \cos \alpha - \frac{1}{\alpha^2}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 x \sin(\alpha x) dx &= \left[-\frac{x}{\alpha} \cos(\alpha x)\right]_0^1 + \int_0^1 \frac{1}{\alpha} \cos(\alpha(x) dx \\
&= \left[-\frac{x}{\alpha} \cos(\alpha x)\right]_0^1 + \left[\frac{1}{\alpha^2} \sin \alpha x\right]_0^1 \\
&= -\frac{1}{\alpha} \cos \alpha + \frac{1}{\alpha^2} \sin \alpha
\end{aligned}$$

All together we have:

$$\begin{aligned}
\mathfrak{F}(f)(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\alpha} \sin \alpha + \frac{1}{\alpha^2} \cos \alpha - \frac{1}{\alpha^2} + i \frac{1}{\alpha} \cos \alpha - i \frac{1}{\alpha^2} \sin \alpha\right)
\end{aligned}$$

Exercise 6 *Find the Fourier transform of*

$$f(x) = \begin{cases} \sin(x) & \textit{if } 0 \leq x \leq 2\pi \\ 0 & \textit{otherwise} \end{cases}$$

Solution 6 *We try to perform integration by parts*

$$\mathcal{F}(f(x))(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin x e^{-i\alpha x} dx$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\left[\sin x \frac{e^{-i\alpha x}}{-i\alpha}\right]_0^{2\pi} + \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \frac{\cos x}{i\alpha} e^{-\alpha i x} dx \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \frac{\cos x}{i\alpha} e^{-\alpha i x} dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{\cos x}{a^2} e^{-i\alpha x}\right]_0^{2\pi} + \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \frac{\sin x}{\alpha^2} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}\alpha^2}\left(e^{-2\pi i\alpha} - 1\right) + \frac{1}{\alpha^2}\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \sin x e^{-i\alpha x} dx
\end{aligned}
$$

It seems we have obtained the term that we started with. However,

$$
\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \sin x e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}\alpha^2}\left(e^{-2\pi i\alpha} - 1\right) + \frac{1}{\alpha^2}\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \sin x e^{-i\alpha x} dx
$$

can be rearranged to

$$
\left(1 - \frac{1}{\alpha^2}\right)\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \sin x e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}\alpha^2}\left(e^{-2\pi i\alpha} - 1\right).
$$

It follows that

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \sin x e^{-i\alpha x} dx &= \left(1 - \frac{1}{\alpha^2}\right)^{-1} \frac{1}{\sqrt{2\pi}\alpha^2}\left(e^{-2\pi i\alpha} - 1\right) \\
&= \left(\frac{\alpha^2 - 1}{\alpha^2}\right)^{-1} \frac{\left(e^{-2\pi i\alpha} - 1\right)}{\sqrt{2\pi}\alpha^2} \\
&= \frac{\alpha^2}{\alpha^2 - 1}\frac{\left(e^{-2\pi i\alpha} - 1\right)}{\alpha^2\sqrt{2\pi}} \\
&= \frac{\left(e^{-2\pi i\alpha} - 1\right)}{\sqrt{2\pi}\left(\alpha^2 - 1\right)}
\end{aligned}
$$

which is the desired Fourier transform.

Exercise 7 (Extra) *We have introduced the Fourier transform*

$$
\mathfrak{F}(f)(\alpha) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx
$$

Different authors define the Fourier transform alternatively by:

$$
\mathfrak{F}_2(f)(\xi) := \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx, \quad \mathfrak{F}_3(f)(\omega) := \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.
$$

Express $\mathfrak{F}_2(f)$ and $\mathfrak{F}_3(f)$ in terms of $\mathfrak{F}(f)$.

Solution 7 *Obviously,*

$$\mathfrak{F}_3(f)(\omega) = \sqrt{2\pi}\mathfrak{F}(f)(\omega).$$

For the other transformation, we write:

$$\mathfrak{F}_2(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}dx = \sqrt{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}dx = \sqrt{2\pi} \cdot \mathfrak{F}(f)(2\pi\xi).$$