Analysis III - 203(d)

Winter Semester 2024

Session 11: November 28, 2024

Exercise 1 *Compute the Fourier coefficients of the following functions, which have period $T = 2\pi$ and have the given values over the interval $[0, 2\pi)$:*

- $$f(x) = e^{x-\pi}$$
- $$g(x) = (x-\pi)^3$$
- $$h(x) = \begin{cases} \sin(x) & 0 \leq x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq x < \frac{3\pi}{2} \\ \sin(x) & \frac{3\pi}{2} \leq x < 2\pi \end{cases}$$

Hint: Recall that $\cos(\frac{\pi}{2} - \theta) = \sin\theta$ and $\cos(\frac{3\pi}{2} - \theta) = -\sin\theta$.

Solution 1

- *To compute the Fourier coefficients of $f(x)$ with a period of 2π, use the formulas from the lecture:*

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} e^{x-\pi}dx = \frac{1}{2\pi}\left[e^{x-\pi}\right]_0^{2\pi} = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)$$

For other coefficients, we apply integration by parts twice, and see that we obtain the same integral again:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\cos(nx)dx \\ &= \frac{1}{\pi}\left[e^{x-\pi}\frac{\sin(nx)}{n}\right]_{x=0}^{x=2\pi} - \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\frac{\sin(nx)}{n}dx \\ &= \frac{1}{\pi}\left[e^{x-\pi}\frac{\sin(nx)}{n}\right]_{x=0}^{x=2\pi} - \frac{1}{\pi}\left[e^{x-\pi}\frac{-\cos(nx)}{n^2}\right]_{x=0}^{x=2\pi} \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\frac{-\cos(nx)}{n^2}dx \\ &= \frac{1}{\pi}\left(e^{\pi}\frac{\sin(2\pi x)}{n} - e^{0-\pi}\frac{\sin(n0)}{n}\right) - \frac{1}{\pi}\left(e^{\pi}\frac{-\cos(n2\pi)}{n^2} - e^{-\pi}\frac{-\cos(n0)}{n^2}\right) \\ &\qquad + \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\frac{-\cos(nx)}{n^2}dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\frac{e^{\pi}-e^{-\pi}}{n^2} + \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\frac{-\cos(nx)}{n^2}dx \\
&= \frac{1}{\pi}\frac{e^{\pi}-e^{-\pi}}{n^2} - \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\frac{\cos(nx)}{n^2}dx
\end{aligned}
$$

We rearrange that

$$
\begin{aligned}
&& \int_0^{2\pi} e^{x-\pi}\cos(nx)dx &= \frac{e^{\pi}-e^{-\pi}}{n^2} - \frac{1}{n^2}\int_0^{2\pi} e^{x-\pi}\cos(nx)dx \\
\implies && \left(1+\frac{1}{n^2}\right)\int_0^{2\pi} e^{x-\pi}\cos(nx)dx &= \frac{e^{\pi}-e^{-\pi}}{n^2} \\
\implies && \int_0^{2\pi} e^{x-\pi}\cos(nx)dx &= \left(1+\frac{1}{n^2}\right)^{-1}\frac{e^{\pi}-e^{-\pi}}{n^2}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
a_n := \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\cos(nx)dx &= \left(1+\frac{1}{n^2}\right)^{-1}\frac{e^{\pi}-e^{-\pi}}{\pi n^2} \\
&= \left(\frac{n^2+1}{n^2}\right)^{-1}\frac{e^{\pi}-e^{-\pi}}{\pi n^2} \\
&= \frac{n^2}{1+n^2}\frac{e^{\pi}-e^{-\pi}}{\pi n^2} \\
&= \frac{e^{\pi}-e^{-\pi}}{(1+n^2)\pi}
\end{aligned}
$$

The coefficients of the sine modes are:

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\sin(nx)dx \\
&= \frac{1}{\pi}\left[-e^{x-\pi}\frac{\cos(nx)}{n}\right]_{x=0}^{x=2\pi} + \frac{1}{\pi}\int_0^{2\pi}\frac{e^{x-\pi}\cos(nx)}{n}dx \\
&= \frac{1}{\pi}\left[-e^{x-\pi}\frac{\cos(nx)}{n}\right]_{x=0}^{x=2\pi} + \frac{1}{\pi}\left[e^{x-\pi}\frac{\sin(nx)}{n^2}\right]_{x=0}^{x=2\pi} - \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\frac{\sin(nx)}{n^2}dx \\
&= \frac{1}{\pi}\frac{e^{-\pi}-e^{\pi}}{n} - \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\frac{\sin(nx)}{n^2}dx
\end{aligned}
$$

Similar as above, it follows that

$$
b_n := \frac{1}{\pi}\int_0^{2\pi} e^{x-\pi}\sin(nx)dx = \frac{1}{1+\frac{1}{n^2}}\frac{1}{\pi}\frac{e^{-\pi}-e^{\pi}}{n} = \frac{1}{\pi}\frac{e^{-\pi}-e^{\pi}}{n+\frac{1}{n}}
$$

- *We compute the average*

$$\frac{a_0}{2} := \frac{1}{2\pi}\int_0^{2\pi} (x-\pi)^3 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} z^3 dz = 0.$$

We compute the coefficients of the cosine modes: for $n \geq 1$, *we find*

$$\begin{aligned} a_n &:= \frac{2}{2\pi}\int_0^{2\pi} (x-\pi)^3 \cos(nx) dx \\ &= \frac{1}{\pi}\int_0^{2\pi} (x-\pi)^3 \cos(nx) dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} z^3 \cos(n(z+\pi)) dz \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} z^3 \cos(nz+n\pi)) dz \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} z^3 \cos(nz)) dz \cdot \begin{cases} 1 & \textit{if } n \textit{ is even} \\ -1 & \textit{if } n \textit{ is odd} \end{cases}. \end{aligned}$$

However, $z^3 \cos(nz)$ *is the product of an odd and an even function, and so the integral vanishes. Explicitly:*

$$\begin{aligned} \int_{-\pi}^{\pi} z^3 \cos(nz)) dz &= \int_{-\pi}^{0} z^3 \cos(nz)) dz + \int_0^{\pi} z^3 \cos(nz)) dz \\ &= -\int_0^{\pi} w^3 \cos(nw)) dw + \int_0^{\pi} z^3 \cos(nz)) dz = 0. \end{aligned}$$

Finally, we compute the coefficients of the sine modes:

$$\begin{aligned} b_n &:= \frac{2}{2\pi}\int_0^{2\pi} (x-\pi)^3 \sin(nx) dx \\ &= \frac{1}{\pi}\int_0^{2\pi} (x-\pi)^3 \sin(nx) dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} z^3 \sin(n(z+\pi)) dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} z^3 \sin(nz) dz \cdot \begin{cases} 1 & \textit{if } n \textit{ is even} \\ -1 & \textit{if } n \textit{ is odd} \end{cases}. \end{aligned}$$

In either case, since z^3 *and* $\sin(nz)$ *are odd functions,*

$$\int_{-\pi}^{\pi} z^3 \sin(nz) dz = 2\int_0^{\pi} z^3 \sin(nz) dz.$$

We perform integration by parts several times:

$$
\begin{aligned}
&\int_0^{\pi} z^3 \sin(nz)dz \\
&= -\frac{1}{n}\left[z^3\cos(nz)\right]_{z=0}^{z=\pi} + \frac{3}{n}\int_0^{\pi} z^2\cos(nz)dz \\
&= -\frac{1}{n}\left[z^3\cos(nz)\right]_{z=0}^{z=\pi} + \frac{3}{n^2}\left[z^2\sin(nz)\right]_{z=0}^{z=\pi} - \frac{6}{n^2}\int_0^{\pi} z\sin(nz)dz \\
&= -\frac{1}{n}\left[z^3\cos(nz)\right]_{z=0}^{z=\pi} + \frac{3}{n^2}\left[z^2\sin(nz)\right]_{z=0}^{z=\pi} + \frac{6}{n^3}\left[z\cos(nz)\right]_{z=0}^{z=\pi} - \frac{6}{n^3}\int_0^{\pi}\cos(nz)dz \\
&= -\frac{1}{n}\left[z^3\cos(nz)\right]_{z=0}^{z=\pi} + \frac{3}{n^2}\left[z^2\sin(nz)\right]_{z=0}^{z=\pi} + \frac{6}{n^3}\left[z\cos(nz)\right]_{z=0}^{z=\pi} + \frac{6}{n^4}\left[\sin(nz)\right]_{z=0}^{z=\pi} \\
&= -\frac{\pi^3(-1)^n}{n} + \frac{6\pi(-1)^n}{n^3} \\
&\implies b_n = \left(\begin{cases} \frac{2\pi^2}{n} - \frac{12}{n^3}, & \text{if } n \text{ is odd,} \\ -\frac{2\pi^2}{n} + \frac{12}{n^3}, & \text{if } n \text{ is even,}\end{cases}\right)\cdot\left(\begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even}\end{cases}\right) = -\frac{2\pi^2}{n} + \frac{12}{n^3}.
\end{aligned}
$$

- *The average of $h(x)$ over 1 period T is 0 therefore:*

$$\frac{a_0}{2} = 0$$

Moreover $h(x)$ is odd therefore $a_n = 0$. Explicitly, we compute

$$a_n = \frac{1}{\pi}\int_0^{2\pi} h(x)\cos(nx)dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sin(x)\cos(nx)dx + \frac{1}{\pi}\int_{\frac{3\pi}{2}}^{2\pi}\sin(x)\cos(nx)dx.$$

To compute the integrals, we apply the product-to-sum identity

$$\sin(x)\cos(nx) = \frac{1}{2}\left(\sin((1+n)x) + \sin((1-n)x)\right).$$

We obtain for the first integral

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sin(x)\cos(nx)dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{2}\left(\sin((1+n)x) + \sin((1-n)x)\right)dx \\
&= \frac{1}{\pi}\left(\frac{1}{2}\left[-\frac{1}{1+n}\cos((1+n)x) - \frac{1}{1-n}\cos((1-n)x)\right]_0^{\frac{\pi}{2}}\right) \\
&= -\frac{1}{n+1}\cos\left(\frac{(n+1)\pi}{2}\right) - \frac{1}{1-n}\cos\left(\frac{(1-n)\pi}{2}\right) + \frac{1}{n+1} + \frac{1}{1-n} \\
&= \frac{1}{2\pi(n^2-1)}\left[(1-n)\cos\left(\frac{\pi}{2}+\frac{n\pi}{2}\right) + (n+1)\cos\left(\frac{\pi}{2}-\frac{n\pi}{2}\right)\right] - \frac{1}{\pi(n^2-1)}
\end{aligned}
$$

$$= \frac{n\sin\left(\frac{n\pi}{2}\right) - 1}{\pi(n^2-1)},$$

where we have applied the hint in the last equation. For the second integral, we obtain

$$\begin{aligned}
&\frac{1}{\pi}\int_{\frac{3\pi}{2}}^{2\pi} \sin(x)\cos(nx)dx = \frac{1}{\pi}\int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2}\left(\sin((1+n)x) + \sin((1-n)x)\right)dx \\
&= \frac{1}{\pi}\left(\frac{1}{2}\left[-\frac{1}{1+n}\cos((1+n)x) - \frac{1}{1-n}\cos((1-n)x)\right]_{\frac{3\pi}{2}}^{2\pi}\right) \\
&= \frac{1}{\pi(n^2-1)} + \frac{1}{2\pi(1-n^2)}\left[(1-n)\cos\left((1+n)\frac{3\pi}{2}\right) + (1+n)\cos\left((1-n)\frac{3\pi}{2}\right)\right] \\
&= \frac{1}{\pi(n^2-1)} + \frac{1}{2\pi(1-n^2)}\left[(1-n)\sin\left(\frac{3\pi n}{2}\right) - (1+n)\sin\left(\frac{3\pi n}{2}\right)\right] \\
&= \frac{1 + n\sin\left(\frac{3\pi n}{2}\right)}{\pi(n^2-1)},
\end{aligned}$$

again by using the hint. Therefore, we have

$$a_n = \frac{n}{\pi(n^2-1)}\left[\sin\left(\frac{\pi n}{2}\right) + \sin\left(\frac{3\pi n}{2}\right)\right] = 0,$$

since the sum-to-product formula for the sine yields

$$\sin\left(\frac{\pi n}{2}\right) + \sin\left(\frac{3\pi n}{2}\right) = 2\sin(2\pi n)\cos(\pi n) = 0.$$

For b_n *we have:*

$$\begin{aligned}
b_n &= \frac{2}{2\pi}\int_0^{\frac{\pi}{2}} \sin(x)\sin(nx)dx + \frac{2}{2\pi}\int_{\frac{3\pi}{2}}^{2\pi} \sin(x)\sin(nx)dx \\
&= \frac{2}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x)\sin(nx)dx \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos((n-1)x) - \cos((n+1)x)dx \\
&= \frac{1}{\pi}\left[\frac{\sin((n-1)x)}{n-1} - \frac{\sin((n+1)x)}{n+1}\right]_{x=0}^{x=\frac{\pi}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n\left(\cos(\frac{n\pi}{2}) + \cos(\frac{3n\pi}{2})\right)}{\pi(1-n^2)} \\
&= \frac{2n\cos(n\pi)\cos(-\frac{n\pi}{2})}{\pi(1-n^2)} = \frac{2n(-1)^n\cos(\frac{n\pi}{2})}{\pi(1-n^2)},
\end{aligned}
$$

where we have again used a reflection formula an product-to-sum formula to simplify the trigonometric expressions. Now we see immediately, that if n is odd, then $b_n = 0$ since $\cos(\frac{n\pi}{2}) = 0$. Whenever n is even, i.e. $n = 2k$ for some $k \in \mathbb{N}$, we have

$$b_n = b_{2k} = \frac{4k(-1)^k}{\pi(1-4k^2)}$$

Exercise 2 *Compute the Fourier coefficients of the function f with period $T = 2\pi$ and which satisfies*

$$f(x) = \sin(x) \text{ if } 0 \le x < \pi,$$

and

$$f(x) = f(-x) \text{ for all } x \in \mathbb{R}.$$

Solution 2 *Note that $f(x)$ is an even function. Using a change of variables, we compute*

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{2}{T}\int_{-\pi}^{0} f(x)\sin(nx)dx + \frac{2}{T}\int_{0}^{\pi} f(x)\sin(nx)dx \\
&= (-1)\frac{2}{T}\int_{\pi}^{0} f(-x)\sin(-nx)dx + \frac{2}{T}\int_{0}^{\pi} f(x)\sin(nx)dx \\
&= \frac{2}{T}\int_{0}^{\pi} f(x)\sin(-nx)dx + \frac{2}{T}\int_{0}^{\pi} f(x)\sin(nx)dx \\
&= -\frac{2}{T}\int_{0}^{\pi} f(x)\sin(nx)dx + \frac{2}{T}\int_{0}^{\pi} f(x)\sin(nx)dx = 0.
\end{aligned}
$$

Therefore $b_n = 0$. The average is

$$\frac{a_0}{2} = \frac{2}{2\pi}\int_0^{\pi} \sin x dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}.$$

The cosine mode coefficients are:

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{4}{T}\int_{0}^{\pi} \sin x\cos(nx)dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4}{T}\int_0^{\pi} \frac{1}{2}\sin((1-n)x) + \frac{1}{2}\sin((n+1)x)dx \\
&= \frac{2}{T}\int_0^{\pi} \sin((1-n)x) + \sin((n+1)x)dx \\
&= \frac{2}{T}\int_0^{\pi} -\sin((n-1)x) + \sin((n+1)x)dx \\
&= \frac{2}{T}\left[\frac{\cos((n-1)x)}{(n-1)}\right]_{x=0}^{x=\pi} + \frac{2}{T}\left[-\frac{\cos((n+1)x)}{(n+1)}\right]_{x=0}^{x=\pi} \\
&= \frac{2}{T}\left(\frac{\cos((n-1)\pi)}{(n-1)} - \frac{1}{(n-1)} - \frac{\cos((n+1)\pi)}{(n+1)} + \frac{1}{(n+1)}\right) \\
&= \frac{2}{T}\left(\frac{\cos(n\pi-\pi)}{(n-1)} - \frac{1}{(n-1)} - \frac{\cos(n\pi+\pi)}{(n+1)} + \frac{1}{(n+1)}\right) \\
&= \frac{2}{T}\left(-\frac{\cos(n\pi)}{(n-1)} - \frac{1}{(n-1)} + \frac{\cos(n\pi)}{(n+1)} + \frac{1}{(n+1)}\right) \\
&= \frac{2}{T}\left(-\frac{(-1)^n}{(n-1)} - \frac{1}{(n-1)} + \frac{(-1)^n}{(n+1)} + \frac{1}{(n+1)}\right) \\
&= \frac{2}{T}\left(\frac{1+(-1)^n}{(n+1)} - \frac{1+(-1)^n}{(n-1)}\right) \\
&= \frac{1}{\pi}\left(\frac{1+(-1)^n}{(n+1)} - \frac{1+(-1)^n}{(n-1)}\right).
\end{aligned}
$$

Here, if n *is odd, the coefficient numerators are* $1+(-1)=0$*, and if* n *is even, they are both equal* 2*. Thus:*

$$
a_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ -\frac{2}{\pi(n-1)} + \frac{2}{\pi(n+1)}, & \text{if } n \text{ is even,} \end{cases}
$$

Exercise 3 *Use Dirichlet's theorem to explain whether the Fourier series converges at* $x \in [0,T]$ *and to which value:*

- $$
f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2-x & \text{if } 1 \le x < 2 \end{cases}, \qquad T = 2.
$$

- $$
f(x) = \begin{cases} \pi & \text{if } 0 \le x < 1 \\ e^{x-1} & \text{if } 1 \le x < 2 \\ \sin(x) & \text{if } 2 \le x < 3 \end{cases}, \qquad T = 3.
$$

Here, the functions have the given period T*.*

Solution 3

- *f is continuous and finite on the closed interval $[0,2]$ therefore the Fourier series, $F_N f$, converges to f when we take the limit of N going to ∞.*
- *f is continuous and finite on the closed intervals $[0,1]$, $[1,2]$ and $[2,3]$. However, note that f is not continuous in 0, 1 and 2. Therefore the Fourier series, Ff, converges to:*

$$g(x) = \begin{cases} \frac{\sin(3)+\pi}{2} & \text{if } x = 0 \\ \pi & \text{if } 0 < x < 1 \\ \frac{\pi+1}{2} & \text{if } x = 1 \\ e^{x-1} & \text{if } 1 < x < 2 \\ \frac{\sin(2)+e^1}{2} & \text{if } x = 2 \\ \sin(x) & \text{if } 2 < x < 3 \\ \frac{\sin(3)+\pi}{2} & \text{if } x = 3 \end{cases}$$

and we extend that function g to a periodic function with period $T = 3$.

Exercise 4 *Give the Fourier series in complex notation, when f has period $T = 2$ and*

$$f(x) = x \text{ for } 0 \leq x < 2.$$

Solution 4 *We can re-use the results about the sawtooth wave.*

$$g(x) = x \text{ for } 0 \leq x < 1$$
$$a_0^g = 1, \quad a_n^g = 0, \quad b_n^g = \frac{-1}{\pi n}$$

Moreover we can write f in terms of g by introducing $x = \frac{y}{2}$

$$f(y) = 2g(y/2)$$

which gives us the Fourier coefficients of f

$$a_0^f = 2, \quad a_n^f = 0, \quad b_n^f = \frac{-2}{\pi n}$$

Using this we can obtain the Fourier coefficients c_n:

$$c_0 = a_0 = 2, \quad c_n = \frac{a_n}{2} - \frac{b_n}{2} i = \frac{1}{n\pi} i, \quad c_{-n} = \frac{a_n}{2} + \frac{b_n}{2} i = -\frac{1}{n\pi} i$$

such that

$$Ff(x) = \sum_{n=-\infty}^{\infty} c_n e^{\pi n x i}$$

Exercise 5 *Find the Fourier coefficients of $\cos(x)^8$ with period $T = 2\pi$. Use Parseval's identity to compute $\int_0^{2\pi} \cos(x)^{16} dx$.*

Solution 5 *We will not use the standard approach in integrating the function f. Instead we try to rewrite f as sum of $\cos(nx)$ using trigonometric formulas:*

$$\begin{aligned}
\cos^2\alpha &= \frac{1}{2} + \frac{1}{2}\cos 2\alpha \\
\cos\alpha\cos\beta &= \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)
\end{aligned}$$

Using these formulas we can rewrite f as follows:

$$\begin{aligned}
\cos^8 x &= (\cos^2 x)^4 \\
&= (\frac{1}{2} + \frac{1}{2}\cos 2x)^4 \\
&= ((\frac{1}{2} + \frac{1}{2}\cos 2x)^2)^2 \\
&= (\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x)^2 \\
&= (\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x)^2 \\
&= (\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x)(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x) \\
&= \frac{1}{16} + \frac{1}{4}\cos 2x + \frac{3}{8}\cos^2 2x + \frac{1}{4}\cos^3 2x + \frac{1}{16}\cos^4 2x \\
&= \frac{1}{16} + \frac{1}{4}\cos 2x + \frac{3}{8}(\frac{1}{2} + \frac{1}{2}\cos 4x) + \frac{1}{4}\cos^3 2x + \frac{1}{16}\cos^4 2x \\
&= \frac{4}{16} + \frac{1}{4}\cos 2x + \frac{3}{16}\cos 4x + \frac{1}{4}\cos^3 2x + \frac{1}{16}\cos^4 2x
\end{aligned}$$

Now we consider $\cos^3 2x$ and $\cos^4 2x$ seperately:

$$\begin{aligned}
\cos^3 2x &= \cos 2x\cos^2 2x \\
&= (\frac{1}{2} + \frac{1}{2}\cos 4x)\cos 2x \\
&= \frac{1}{2}\cos 2x + \frac{1}{2}\cos 4x\cos 2x \\
&= \frac{1}{2}\cos 2x + \frac{1}{4}\cos 6x + \frac{1}{4}\cos 2x \\
&= \frac{3}{4}\cos 2x + \frac{1}{4}\cos 6x
\end{aligned}$$

$$\cos^4 2x = (\cos^2 2x)^2$$

$$
\begin{aligned}
&= (\frac{1}{2} + \frac{1}{2}\cos 4x)^2 \\
&= \frac{1}{4} + \frac{1}{2}\cos 4x + \frac{1}{4}\cos^2 4x \\
&= \frac{1}{4} + \frac{1}{2}\cos 4x + \frac{1}{4}(\frac{1}{2} + \frac{1}{2}\cos 8x) \\
&= \frac{3}{8} + \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x
\end{aligned}
$$

putting everything together we have:

$$\cos^8 x = \frac{35}{128} + \frac{7}{16}\cos 2x + \frac{7}{32}\cos 4x + \frac{1}{16}\cos 6x + \frac{1}{128}\cos 8x$$

For the second part we use Parseval's identity which is given by:

$$\frac{2}{T}\int_0^T f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=0}^{\infty}(a_n)^2 + (b_n)^2$$

substituting our Fourier coefficients and f gives:

$$
\begin{aligned}
\frac{2}{T}\int_0^T f(x)^2 dx &= \frac{a_0^2}{2} + \sum_{n=0}^{\infty}(a_n)^2 + (b_n)^2 \\
\frac{1}{\pi}\int_0^{2\pi}\cos(x)^{16} dx &= 2\left(\frac{35}{128}\right)^2 + \left(\frac{7}{16}\right)^2 + \left(\frac{7}{32}\right)^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{128}\right)^2. \\
\implies \int_0^{2\pi}\cos(x)^{16} dx &= \pi\left(2\left(\frac{35}{128}\right)^2 + \left(\frac{7}{16}\right)^2 + \left(\frac{7}{32}\right)^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{128}\right)^2\right)
\end{aligned}
$$

We simplify this value:

$$2\left(\frac{35}{128}\right)^2 + \left(\frac{7}{16}\right)^2 + \left(\frac{7}{32}\right)^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{128}\right)^2 = 6435$$

Therefore, in total,

$$\int_0^{2\pi}\cos(x)^{16} dx = \pi\frac{6435}{16384} \approx 1.233895796$$

Exercise 6 *Give the Fourier series of the function f with period $T = 2$ and*

$$f(x) = \cos(x) \text{ for } -1 \le x < 1.$$

Give the Fourier series in standard form and in complex notation. Compare $F_3 f$ at the points $x = -\pi/4, 0, \pi/4$ with the original function f.

Solution 6 *We calculate the complex Fourier coefficients*

$$c_0 = \frac{1}{2}\int_{-1}^{1} \cos x dx = \frac{1}{2}[\sin x]_{-1}^{1} = \sin 1$$

$$\begin{aligned}
c_n &= \frac{1}{2}\int_{-1}^{1} \cos x e^{-\pi n x i} dx \\
&= \frac{1}{2}\int_{-1}^{1} \left(\frac{1}{2}e^{xi} + \frac{1}{2}e^{-xi}\right) e^{-\pi n x i} dx \\
&= \frac{1}{2}\int_{-1}^{1} \left(\frac{1}{2}e^{(1-n\pi)xi} + \frac{1}{2}e^{-(1+n\pi)xi}\right) dx \\
&= \frac{1}{2}\left[\frac{1}{(1-n\pi)i}\frac{1}{2}e^{(1-n\pi)xi} - \frac{1}{(1+n\pi)i}\frac{1}{2}e^{-(1+n\pi)xi}\right]_{-1}^{1} \\
&= \frac{1}{2(1-n\pi)i}\left(\frac{1}{2}e^{(1-n\pi)i} - \frac{1}{2}e^{-(1-n\pi)i}\right) - \frac{1}{2(1+n\pi)i}\left(\frac{1}{2}e^{(1+n\pi)i} + \frac{1}{2}e^{-(1+n\pi)i}\right) \\
&= \frac{1}{2(1-n\pi)}\sin(1-\pi n) + \frac{1}{2(1+n\pi)}\sin(1+\pi n)
\end{aligned}$$

Using c_n we can determine a_n and b_n

$$\frac{a_0}{2} = c_0 = \sin 1, \quad a_n = c_n + c_{-n} = \frac{1}{(1-n\pi)}\sin(1-\pi n) + \frac{1}{(1+n\pi)}\sin(1+\pi n),$$
$$b_n = i(c_n - c_{-n}) = 0.$$

The Fourier series in standard and in complex notation are given by:

$$Ff(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2(1-n\pi)}\sin(1-\pi n) + \frac{1}{2(1+n\pi)}\sin(1+\pi n)\right) e^{\frac{2\pi n x}{T} i}$$
$$Ff(x) = \sin(1) + \sum_{n=1}^{\infty} \left(\frac{1}{(1-n\pi)}\sin(1-\pi n) + \frac{1}{(1+n\pi)}\sin(1+\pi n)\right) \cos(\pi n x)$$

We compare $F_3 f(x)$ and $f(x)$ at several points

$$F_3 f(0) = 1.00664, \qquad F_3 f\left(-\frac{\pi}{4}\right) \approx 0.69196, \qquad F_3 f\left(\frac{\pi}{4}\right) \approx 0.69196,$$
$$f(0) = 1, \qquad f\left(-\frac{\pi}{4}\right) \approx 0.70711, \qquad f\left(\frac{\pi}{4}\right) \approx 0.70711.$$