# Analysis III - 203(d)

Winter Semester 2024

## Session 8: November 7, 2024

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**Exercise 1** *Consider the surface*

$$S := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \|x\| = 1, x_3 > 0 \right\}$$

- *Find a parameterization of $S$ and find the unit normal corresponding to that parameterization.*
- *Find a parameterization of $C = \partial S$.*
- *Verify Stokes theorem with the vector field*

$$\vec{F}(x_1, x_2, x_3) = (-x_2, x_1, x_3).$$

**Solution 1**

- $$\Phi : \Omega := [0, 2\pi) \times [0, \frac{\pi}{2}) \mapsto S : (\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

$$\vec{n} = \frac{\partial_\theta \Phi \times \partial_\phi \Phi}{||\partial_\theta \Phi \times \partial_\phi \Phi||} = \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix}$$

- *We first parameterize the boundary of the parameter space $\Omega$*

$$\gamma : [0, 2\pi) \mapsto \partial\Omega : \theta \mapsto (\theta, \frac{\pi}{2}),$$

*which gives us the parameterization of the boundary as follows:*

$$\Phi \circ \gamma : [0, 2\pi) \mapsto \partial C : \theta \mapsto (\cos\theta, \sin\theta, 0).$$

- *We have to show that:*

$$\iint_S \nabla \times \vec{F} d\sigma = \oint_C \vec{F} d\ell$$

$$\begin{aligned} \iint_S \nabla \times \vec{F} d\sigma &= \iint_S \nabla \times \vec{F} \cdot \vec{n} ||\partial_\theta \Phi \times \partial_\phi \Phi|| d\sigma \\ &= \iint_S \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix} \sin\phi d\sigma \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2\cos\phi \sin\phi d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin 2\phi d\phi d\theta \\
&= 2\pi \left[ -\frac{1}{2}\cos 2\phi \right]_0^{\frac{\pi}{2}} = 2\pi
\end{aligned}
$$

$$
\begin{aligned}
\oint_C \vec{F} d\ell &= \int_0^{2\pi} \vec{F}(\Phi \circ \gamma) \cdot \frac{d(\Phi \circ \gamma)}{d\theta} d\theta \\
&= \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\phi \\ 0 \end{pmatrix} d\theta \\
&= \int_0^{2\pi} 1 d\theta = 2\pi
\end{aligned}
$$

**Exercise 2** *Let $S$ be the surface with parameterization $\Phi : \Omega \to S$ as follows:*

$$
S := \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 = 1,\ -1 < x_3 < 1 \right\}, \tag{1}
$$

*Suppose we have a vector field*

$$
\vec{F}(x_1, x_2, x_3) := (x_2 x_3, -x_1 x_3, 0). \tag{2}
$$

- *Find a parameterization of the unit normal $\vec{n}$ given associated with that parameterization.*
- *Compute the scalar field $\vec{F} \cdot \vec{n}$ and calculate the integrals of this scalar fields over $S$.*
- *Compute the integral using the formula*

$$
\int_\Omega \vec{F} \cdot (\partial_S \Phi \times \partial_t \Phi) ds dt \tag{3}
$$

  *and compare the result.*
- *Find a counterclockwise parameterization of $\partial\Omega$ and use it build a parameterization of the curve $C = \partial S$.*
- *Compute the integrals*

$$
\oint_C \vec{F} dl. \tag{4}
$$

**Solution 2**

- *In a similar fashion as Exercise 1 Subquestion 2 we obtain the following normal vector:*

$$\vec{n} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

- $$\vec{F} \cdot \vec{n} = \begin{pmatrix} x_2 x_3 \\ -x_1 x_3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 x_2 x_3 - x_1 x_2 x_3 = 0$$

$$\int_S \vec{F} \cdot \vec{n} d\sigma = \int_S 0 d\sigma = 0$$

- *We define the parameterisation:*

$$\Phi : [0, 2\pi) \times [-1, 1] \mapsto S, (\theta, z) \mapsto (\cos\theta, \sin\theta, z)$$

$$\begin{aligned} \int_\Omega \vec{F}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) ds dt &= \int_0^{2\pi} \int_{-1}^{1} \begin{pmatrix} z \sin\theta \\ -z \cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} ds dt \\ &= \int_0^{2\pi} \int_{-1}^{1} 0 ds dt = 0 \end{aligned}$$

- *A counter clockwise parameterisation of $\partial\Omega$ is given by:*

$$\begin{aligned} \gamma_1 &: [0, 2\pi) \mapsto \partial\Omega_1, (\theta) \mapsto (\theta, -1) \\ \gamma_2 &: [-1, 1) \mapsto \partial\Omega_2, (r) \mapsto (2\pi, r) \\ \gamma_3 &: [0, 2\pi) \mapsto \partial\Omega_3, (\theta) \mapsto (2\pi - \theta, 1) \\ \gamma_4 &: [-1, 1) \mapsto \partial\Omega_4, (r) \mapsto (0, -r) \end{aligned}$$

*which we can use to build a parameterization of the curve $C = \partial S$:*

$$\begin{aligned} \Phi \circ \gamma_1 &: [0, 2\pi) \mapsto C_1, (\theta) \mapsto (\cos\theta, \sin\theta, -1) \\ \Phi \circ \gamma_2 &: [1, -1) \mapsto C_2, (r) \mapsto (1, 0, r) \\ \Phi \circ \gamma_3 &: [0, 2\pi) \mapsto C_3, (\theta) \mapsto (\cos\theta, -\sin\theta, 1) \\ \Phi \circ \gamma_4 &: [-1, 1) \mapsto C_4, (r) \mapsto (1, 0, -r) \end{aligned}$$

•

$$
\begin{aligned}
\oint_C \vec{F} dl &= \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} \vec{F} dl \\
&= \sum_{i=1}^{4} \int_{C_i} \vec{F}(\Phi \circ \gamma_i) \cdot (\Phi \circ \gamma_i)' dl \\
&= \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} d\theta + \int_{-1}^{1} \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dr + \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} d\theta + \\
&+ \int_{-1}^{1} \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dr \\
&= 2 \int_0^{2\pi} \sin^2\theta + \cos^2\theta dr \\
&= 2 \int_0^{2\pi} d\theta = 4\pi
\end{aligned}
$$

• $\vec{F}$ *has non-zero curl, so it cannot be a gradient of a scalar field.*

**Exercise 3** *Let $S$ be a surface with parameterization $\Phi : \Omega \to S$ as follows:*

$$
\begin{aligned}
S &:= \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 > 0 \right\}, && (5) \\
\Omega &= \left\{ (s,t) \in \mathbb{R}^2 \;\middle|\; s + t < 1,\ s, t > 0 \right\}, && (6) \\
\Phi &: \Omega \to S, \quad (s,t) \mapsto (s, t, 1 - s - t). && (7)
\end{aligned}
$$

*We consider the vector fields*

$$
\vec{F}(x_1, x_2, x_3) := (x_2 x_3, -x_1 x_3, x_1 x_2), \quad \vec{G}(x_1, x_2, x_3) := (-x_2, x_1, x_3) \tag{8}
$$

- *Find the unit normal $\vec{n}$ given associated with that parameterization.*
- *Compute the scalar fields $\vec{F} \cdot \vec{n}$ and $\vec{G} \cdot \vec{n}$ and calculate the integrals of these scalar fields over $S$.*
- *Compute the integrals using the formula*

$$
\int_\Omega \vec{F} \cdot (\partial_S \Phi \times \partial_t \Phi) ds dt \tag{9}
$$

  *and compare the result.*

- *Find a counterclockwise parameterization of $\partial\Omega$ and use it build a parameterization of the curve $C = \partial S$.*

- *Compute the integrals*

$$\oint_C \vec{F} dl, \quad \oint_C \vec{G} dl. \tag{10}$$

- *Can you exclude that $\vec{F}$ or $\vec{G}$ are gradients of a scalar field?*

**Solution 3**

- *In a similar fashion as exercise 1 subquestion 2 we obtain the following normal vector:*

$$\vec{n} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

-

$$\begin{aligned}
\int_\Omega \vec{F}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) dsdt &= \int_\Omega \vec{F}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) \frac{||\partial_S \Phi \times \partial_t \Phi||}{||\partial_S \Phi \times \partial_t \Phi||} dsdt \\
&= \int_\Omega \vec{F}(\Phi) \cdot \vec{n} ||\partial_S \Phi \times \partial_t \Phi|| dsdt \\
&= \int_S \vec{F} \cdot \vec{n} d\sigma
\end{aligned}$$

-

$$\partial_s \Phi \times \partial_t \Phi = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
\int_S \vec{F}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) dsdt &= \int_0^1 \int_0^{1-t} \begin{pmatrix} t(1-s-t) \\ -s(1-s-t) \\ st \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dsdt \\
&= \int_0^1 \int_0^{1-t} t(1-s-t) - s(1-s-t) + stdsdt
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \int_0^{1-t} t(1-s-t) dsdt = \int_0^1 \frac{1}{2} t - t^2 + \frac{1}{2} t^3 dt = \frac{1}{24} \\
&\int_0^1 \int_0^{1-t} s(1-s-t) dsdt = \int_0^1 \int_0^{1-t} \frac{1}{2}(1-s-t)^2 dsdt = \int_0^1 \frac{1}{6}(1-t)^3 dt = \frac{1}{24}
\end{aligned}$$

$$\int_0^1 \int_0^{1-t} stdsdt = \int_0^1 \frac{1}{2}(1-t)^2 tdt = \int_0^1 \frac{1}{6}(1-t)^3 dt = \frac{1}{24}$$

$$\int_0^1 \int_0^{1-t} t(1-s-t) - s(1-s-t) + stdsdt = \frac{1}{24} - \frac{1}{24} + \frac{1}{24} = \frac{1}{24}$$

$$\begin{aligned}
\int_S \vec{G}(\Phi) \cdot (\partial_S \Phi \times \partial_t \Phi) dsdt &= \int_0^1 \int_0^{1-t} \begin{pmatrix} -t \\ s \\ 1-s-t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dsdt \\
&= \int_0^1 \int_0^{1-t} 1 - 2tdsdt \\
&= \int_0^1 (1-2t)(1-t) dt \\
&= \left[ -\frac{1}{2}(1-2t)(1-t) \right]_0^1 - \int_0^1 (1-t)^2 dt \\
&= \frac{1}{2} - \left[ -\frac{1}{3}(1-t)^3 \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\end{aligned}$$

- *The boundary of $\Omega$ consists of three lines which can be parameterized in a counter clockwise way as follows:*

$$\begin{aligned}
\gamma_1 &: [0,1] \mapsto \partial\Omega_1, (r) \mapsto (r, 0), \\
\gamma_2 &: [0,1] \mapsto \partial\Omega_2, (r) \mapsto (1-r, r), \\
\gamma_3 &: [0,1] \mapsto \partial\Omega_3, (r) \mapsto (0, 1-r),
\end{aligned}$$

*which we can use to parameterize the curve $C$ which consist of three lines:*

$$\begin{aligned}
\Phi \circ \gamma_1 &: [0,1] \mapsto C_1, (r) \mapsto (r, 0, 1-r), \\
\Phi \circ \gamma_2 &: [0,1] \mapsto C_2, (r) \mapsto (1-r, r, 0), \\
\Phi \circ \gamma_3 &: [0,1] \mapsto C_3, (r) \mapsto (0, 1-r, r),
\end{aligned}$$

- $$\oint_C \vec{F} dl = \oint_{C_1 \cup C_2 \cup C_3} \vec{F} dl$$

$$= \int_{C_1} \vec{F}(\Phi\circ\gamma_1)\cdot\frac{d(\Phi\circ\gamma_1)}{dr}dl + \int_{C_2} \vec{F}(\Phi\circ\gamma_2)\cdot\frac{d(\Phi\circ\gamma_2)}{dr}dl + \int_{C_3} \vec{F}(\Phi\circ\gamma_3)\cdot\frac{d(\Phi\circ\gamma_3)}{dr}dl$$

$$= \int_0^1 \begin{pmatrix} 0 \\ -r+r^2 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} dr + \int_0^1 \begin{pmatrix} 0 \\ 0 \\ r-r^2 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} dr + \int_0^1 \begin{pmatrix} r-r^2 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} dr =$$

$$= 0$$

$$\oint_C \vec{G}dl = \oint_{C_1\cup C_2\cup C_3} \vec{G}dl$$

$$= \int_{C_1} \vec{G}(\Phi\circ\gamma_1)\cdot\frac{d(\Phi\circ\gamma_1)}{dr}dl + \int_{C_2} \vec{G}\Phi\circ\gamma_2)\cdot\frac{d(\Phi\circ\gamma_2)}{dr}dl + \int_{C_3} \vec{G}(\Phi\circ\gamma_3)\cdot\frac{d(\Phi\circ\gamma_3)}{dr}dl$$

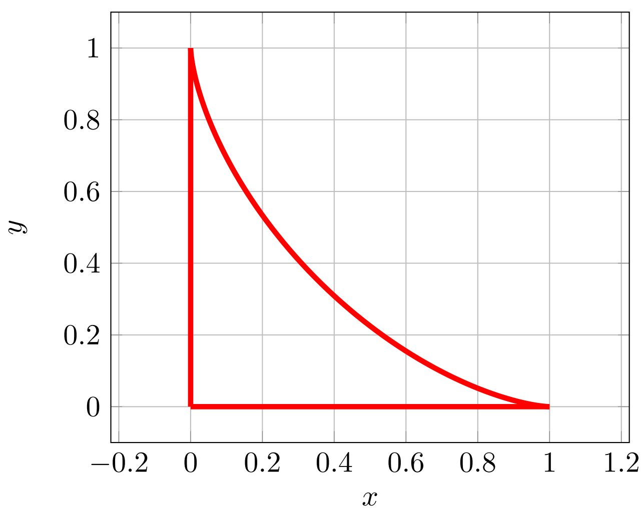
$$= \int_0^1 \begin{pmatrix} 0 \\ r \\ 1-r \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} dr + \int_0^1 \begin{pmatrix} -r \\ 1-r \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} dr + \int_0^1 \begin{pmatrix} r-1 \\ 0 \\ r \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} dr =$$

$$= \int_0^1 r - 1dr + \int_0^1 r + 1 - rdr + \int_0^1 rdr$$

$$= \int_0^1 2rdr = 1$$

- *Both $\vec{F}$ and $\vec{G}$ have non-zero curl, so they cannot be gradients of a scalar field.*

**Exercise 4 (Green's theorem)** *Consider the following closed curve $C$:*




*The curved part of the curve can be parameterized by*

$$\gamma : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^2, \quad t \mapsto \left(\cos^3(t), \sin^3(t)\right).$$

- *Find parameterizations of the other two curves.*
- *With the $\vec{F}(x_1, x_2) = (x_2, -x_1)$ compute the curve integral*

$$\oint_C \vec{F} dl$$

- *Use Green's theorem to explain how to compute the size of the area enclosed by the curve $C$.*

**Solution 4**

- $$\gamma_1 : [0,1] \to \mathbb{R}^2, \quad t \mapsto (0, 1-t)\,.$$

  $$\gamma_2 : [0,1] \to \mathbb{R}^2, \quad t \mapsto (t, 0)\,.$$

- $$\begin{aligned}
\oint_C \vec{F} dl &= \int_0^1 \begin{pmatrix} 1-t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} 0 \\ -t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \int_0^{\frac{\pi}{2}} \begin{pmatrix} \sin^3 t \\ -\cos^3 t \end{pmatrix} \cdot \begin{pmatrix} -3\cos^2 t \sin t \\ 3\sin^2 t \cos t \end{pmatrix} dt \\
&= \int_0^{\frac{\pi}{2}} -3\cos^2 t \sin^2 t dt \\
&= \int_0^{\frac{\pi}{2}} -\frac{3}{4} \sin^2 2t dt \\
&= \int_0^{\frac{\pi}{2}} \frac{3}{4} \left( \frac{1}{2} \cos 4t - \frac{1}{2} \right) dt \\
&= \int_0^{\frac{\pi}{2}} -\frac{3}{8} dt = -\frac{3\pi}{16}
\end{aligned}$$

- *We want to evaluate:*

  $$\iint_\Omega 1 dx dy$$

  *and Green's theorem tells us that:*

  $$\oint_C \vec{F} dl = \iint_\Omega \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} dx_1 dx_2$$

  *In the previous exercise we solved the line integral with the vector field $\vec{F}(x_1, x_2) = (x_2, -x_1)$. If we apply Greens theorem for this vector field then*

  $$-\frac{3\pi}{16} = \oint_C \vec{F} dl = \iint_\Omega \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} dx_1 dx_2 = \iint_\Omega -2 dx_1 dx_2$$

*Therefore we have that*

$$-\frac{3\pi}{16} = \iint_\Omega -2dx_1dx_2$$

*and consequently*

$$\frac{3\pi}{32} = \iint_\Omega dx_1dx_2$$

**Exercise 5 (Stokes's theorem)** *We have a vector field*

$$\vec{F}(x_1, x_2, x_3) = (x_1x_2, x_2x_3, x_1x_3)$$

*and a surface*

$$S := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = 1,\ x_2 \geq 0, x_3 \geq 0\right\}$$

- *Find parameterizations of the surface $S$ and a parameterization of its boundary curve $C$.*
- *Use Stokes' theorem to compute the integral*

$$\iint_S \operatorname{curl} \vec{F}\, d\sigma.$$

**Solution 5** *The surface is parameterized by*

$$\Omega := [0,\pi] \times [0, \frac{\pi}{2}] \mapsto S : (\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$$

*The boundary of domain $S$ is parameterised by two curves*

$$\begin{aligned} \gamma_1 &: [0,\pi] \to \mathbb{R}^3, \quad t \mapsto (\cos t, \sin t, 0), \\ \gamma_2 &: [0,\pi] \to \mathbb{R}^3, \quad t \mapsto (-\cos t, 0, \sin t), \end{aligned}$$

*Note that we do not need $\Phi \circ \gamma_3$ to parameterise the boundary of $S$. Moreover it is not a regular parameterisation.*

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F}\, d\sigma &= \int_0^\pi \begin{pmatrix} \cos t \sin t \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt + \int_0^\pi \begin{pmatrix} 0 \\ 0 \\ -\sin t \cos t \end{pmatrix} \cdot \begin{pmatrix} \sin t \\ 0 \\ \cos t \end{pmatrix} dt \\ &= \int_0^\pi -\cos t \sin^2 t dt - \int_0^\pi \sin t \cos^2 t dt \\ &= -\int_0^0 u^2 du - \int_{-1}^1 v^2 dv \\ &= -\frac{2}{3} \end{aligned}$$