

Analysis III - 203(d)

Winter Semester 2024

Session 7: October 31, 2024

Exercise 1 Verify the divergence theorem for the following vector field $\vec{F}$ and volume V :

$$\vec{F}(x_1, x_2, x_3) := (x_1 x_3, x_2, x_2), \quad V := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 < 1 \}.$$

Note that V is just the three-dimensional unit ball.

Solution 1 We have to show that the following identity holds:

$$\iiint_V \nabla \cdot \vec{F} \, dV = \oint \vec{F} \cdot \vec{n} \, d(\partial V)$$

We start with the volume integral:

$$\begin{aligned} & \iiint_V \nabla \cdot \vec{F} \, dV \\ &= \iiint_V x_3 + 1 \, dV \\ &= \int_0^{2\pi} \int_0^\pi \int_0^1 (r \cos \phi + 1) r^2 \sin \phi \, dr \, d\phi \, d\theta \\ &= 2\pi \int_0^1 r^3 \, dr \int_0^\pi \cos \phi \sin \phi \, d\phi + 2\pi \int_0^1 r^2 \, dr \int_0^\pi \sin \phi \, d\phi \\ &= 2\pi \left[\frac{1}{4} r^4 \right]_0^1 \left[-\frac{1}{4} \cos 2\phi \right]_0^\pi + 2\pi \left[\frac{1}{3} r^3 \right]_0^1 [-\cos \phi]_0^\pi \\ &= 2\pi \left(\frac{1}{4} - 0 \right) \left(\frac{1}{4} - -\frac{1}{4} \right) + 2\pi \left(\frac{1}{3} - 0 \right) (1 - -1) = \frac{4\pi}{3}. \end{aligned}$$

Next we consider the surface integral. We need a parameterisation for the surface. One can consider:

$$\Phi : [0, 2\pi) \times (0, \pi) \mapsto S : (\theta, \phi) \mapsto (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$$

with

$$\partial_\theta \Phi \times \partial_\phi \Phi = \begin{pmatrix} -\sin \theta \sin \phi \\ \cos \theta \sin \phi \\ 0 \end{pmatrix} \times \begin{pmatrix} -\cos \theta \cos \phi \\ \sin \theta \cos \phi \\ -\sin \phi \end{pmatrix} = - \begin{pmatrix} \cos \theta \sin^2 \phi \\ \sin \theta \sin^2 \phi \\ \sin \phi \cos \phi \end{pmatrix}$$

Note this is inward pointing. Therefore we need an extra minus sign.

$$\oint \vec{F} \cdot \vec{n} \, dS$$

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^\pi \begin{pmatrix} \cos \theta \sin \phi \cos \phi \\ \sin \theta \sin \phi \\ \sin \theta \sin \phi \end{pmatrix} \cdot \begin{pmatrix} \cos \theta \sin^2 \phi \\ \sin \theta \sin^2 \phi \\ \sin \phi \cos \phi \end{pmatrix} d\phi d\theta \\
&= \int_0^{2\pi} \cos^2 \theta d\theta \int_0^\pi \sin^3 \phi \cos \phi d\phi + \int_0^{2\pi} \sin^2 \theta d\theta \int_0^\pi \sin^3 \phi d\phi + \int_0^{2\pi} \sin \theta d\theta \int_0^\pi \sin^2 \phi \cos \phi d\phi.
\end{aligned}$$

The last term is zero due to the fact that the integral of $\sin \theta$ over 1 period is equal to 0. The other integrals are evaluated seperately, where we use the substitution $u = \cos \phi$ several times:

$$\begin{aligned}
&\int_0^\pi \sin^3 \phi \cos \phi d\phi \\
&= \int_0^\pi (1 - \cos^2 \phi) \sin \phi \cos \phi d\phi \\
&= \int_0^\pi \cos \phi \sin \phi - \cos^3 \phi \sin \phi d\phi \\
&= \int_0^\pi \frac{1}{2} [\cos^2(\phi)]' + \frac{1}{4} [\cos^4(\phi)]' du \\
&= \frac{1}{2} [\cos^2(\phi)]_{\phi=0}^{\phi=\pi} + \frac{1}{4} [\cos^4(\phi)]_{\phi=0}^{\phi=\pi} = 0
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^{2\pi} \sin^2 \theta d\theta \\
&= \int_0^{2\pi} \frac{1}{2} (1 - \cos 2\theta) d\theta \\
&= \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \pi
\end{aligned}$$

Next,

$$\begin{aligned}
&\int_0^\pi \sin^3 \phi d\phi \\
&= \int_0^\pi (1 - \cos^2 \phi) \sin \phi d\phi \\
&= \int_{-1}^1 1 - u^2 du \\
&= \left[u - \frac{1}{3} u^3 \right]_{-1}^1 = \frac{4}{3}
\end{aligned}$$

If we put everything together we get:

$$\oint \vec{F} \cdot \vec{n} d(\partial\Omega)$$

$$\begin{aligned}
&= \int_0^{2\pi} \cos^2 \theta \, d\theta \int_0^\pi \sin^3 \phi \cos \phi \, d\phi + \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^\pi \sin^3 \phi \, d\phi \\
&= \int_0^{2\pi} \cos^2 \theta \, d\theta \cdot 0 + \pi \frac{4}{3} = \frac{4\pi}{3}.
\end{aligned}$$

We conclude that both integrals give the same value thereby verifying the divergence theorem.

Exercise 2 Consider the volume

$$V := \{\vec{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 < 5\}, \quad (1)$$

- What is the surface S of this volume?
- Find the outward pointing unit normal $\vec{n}$ along the surface S of this volume. Write $\vec{n}$ in terms of (x_1, x_2, x_3) at any point on the surface S .
- Find a parameterization of the surface S .
- Find a vector field $\vec{F}$ such that $\vec{F} \cdot \vec{n} = x_1^2 + x_2 + x_3$ along the surface S .
- Use the divergence theorem to compute

$$\iint_S x_1^2 + x_2 + x_3 dx_1 dx_2 dx_3. \quad (2)$$

Solution 2 • $S := \{\vec{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 5\}$

- We can write $G(x_1, x_2, x_3) = -5 + x_1^2 + x_2^2 + x_3^2$. Then since we know that the gradient is normal to the level curves we have that:

$$\vec{n} = \frac{\nabla G}{\|\nabla G\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- A possible parameterization is:

$$\Phi : [0, 2\pi) \times [0, \pi) \mapsto S, (\theta, \phi) \mapsto (\sqrt{5} \sin \phi \cos \theta, \sqrt{5} \sin \phi \sin \theta, \sqrt{5} \cos \phi)$$

- $\vec{F} = \sqrt{5} \begin{pmatrix} x_1 \\ 1 \\ 1 \end{pmatrix}$

- We use the divergence theorem:

$$\iiint_S x_1^2 + x_2 + x_3 dx_1 dx_2 dx_3 = \oint_S \vec{F} \cdot \vec{n} d\sigma$$

$$\begin{aligned}
&= \iiint_V \nabla \cdot F dV \\
&= \iiint_V \sqrt{5} dV = \sqrt{5} \cdot \iiint_V 1 dV = \frac{4}{3} \pi \sqrt{5}^4 = \frac{4}{3} \pi \cdot 25
\end{aligned}$$

In the last step, we used the formula for the volume of the ball with radius $r = \sqrt{5}$.

Exercise 3 Consider the volume

$$V := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < x_3 < 1 \}$$

Find the the boundary S of this volume and compute its surface area.

Solution 3 The boundary of this domain consist of two surfaces:

$$\begin{aligned}
S_a &= \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1, x_3 = 1 \}, \\
S_b &= \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = x_3, 0 < x_3 < 1 \}
\end{aligned}$$

We have that $\partial V = \partial \Omega_a \cup \partial V_b$. To calculate the area we need a parameterisation of both surfaces:

$$\begin{aligned}
\Phi_a &: [0, 2\pi) \times (0, 1) \mapsto S_a : (\theta, r) \mapsto (r \cos \theta, r \sin \theta, 1), \\
\Phi_b &: [0, 2\pi) \times (0, 1) \mapsto S_b : (\theta, z) \mapsto (\sqrt{z} \cos \theta, \sqrt{z} \sin \theta, z).
\end{aligned}$$

Moreover we have that

$$\|\partial_\theta \Phi_a \times \partial_r \Phi_a\| = \left\| \begin{pmatrix} -r \sin \theta \\ r \cos \theta \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} \right\| = r$$

and

$$\|\partial_\theta \Phi_b \times \partial_z \Phi_b\| = \left\| \begin{pmatrix} -\sqrt{z} \sin \theta \\ \sqrt{z} \cos \theta \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2\sqrt{z}} \cos \theta \\ \frac{1}{2\sqrt{z}} \sin \theta \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{z} \sin \theta \\ \sqrt{z} \cos \theta \\ -\frac{1}{2} \end{pmatrix} \right\| = \sqrt{z + \frac{1}{4}}$$

Now the area is given by:

$$\begin{aligned}
\iint_{\partial \Omega} 1 d(\partial \Omega) &= \iint_{\partial \Omega_a} 1 dS_a + \iint_{\partial \Omega_b} 1 dS_b \\
&= \int_0^{2\pi} \int_0^1 r dr d\theta + \int_0^{2\pi} \int_0^1 \sqrt{z + \frac{1}{4}} dz d\theta \\
&= 2\pi \left[\frac{1}{2} r^2 \right]_0^1 + 2\pi \left[\frac{2}{3} \left(z + \frac{1}{4} \right)^{\frac{3}{2}} \right]_0^1 \\
&= \pi + \frac{4\pi}{3} \left(\left(\frac{5}{4} \right)^{\frac{3}{2}} - \left(\frac{1}{4} \right)^{\frac{3}{2}} \right) = \pi + \pi \frac{5\sqrt{5} - 1}{6} = 5\pi \frac{1 + \sqrt{5}}{6}.
\end{aligned}$$

Exercise 4 Find a regular parameterization $\Phi(s, t)$ of the surface

$$S := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 < x_3 < 1, x_1^2 + x_2^2 = 1 + x_3^2 \}$$

Compute cross product $\partial_s \Phi(s, t) \times \partial_t \Phi(s, t)$ and its norm.

Solution 4 We define the following parameterisation for the surface S :

$$\begin{aligned} \Phi : [0, 2\pi) \times (0, 1) &\mapsto S, \quad (s, t) \mapsto \left(\sqrt{1+t^2} \cos s, \sqrt{1+t^2} \sin s, t \right) \\ \partial_s \Phi(s, t) \times \partial_t \Phi(s, t) &= \begin{pmatrix} -\sqrt{1+t^2} \sin s \\ \sqrt{1+t^2} \cos s \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{-t}{\sqrt{1+t^2}} \cos s \\ \frac{-t}{\sqrt{1+t^2}} \sin s \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1+t^2} \cos s \\ \sqrt{1+t^2} \sin s \\ t \end{pmatrix} \\ \|\partial_s \Phi(s, t) \times \partial_t \Phi(s, t)\| &= \left\| \begin{pmatrix} \sqrt{1+t^2} \cos s \\ \sqrt{1+t^2} \sin s \\ t \end{pmatrix} \right\| = \sqrt{1+2t^2} \end{aligned}$$

Exercise 5 Consider the volume

$$V := \{ \vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 < 1, x_1, x_2, x_3 > 0 \}, \quad (3)$$

Suppose we have a vector field

$$\vec{F}(x_1, x_2, x_3) = (x_1^2 x_2, 3x_2^2 x_3, 9x_3^2 x_1) \quad (4)$$

Use the divergence theorem to compute the surface integral

$$\iint_S \vec{F} d\sigma. \quad (5)$$

Solution 5 Given any scalar function $g : V \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \oint_S \vec{F} d\sigma &= \iiint_V \nabla \cdot \vec{F} dV \\ &= \iiint_V 2x_1 x_2 + 6x_2 x_3 + 18x_3 x_1 dV \\ &= \int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 2x_1 x_2 + 6x_2 x_3 + 18x_3 x_1 dx_1 dx_2 dx_3 \end{aligned}$$

We solve each integral separately. Note that we apply integration by parts in between steps.

$$\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 2x_1 x_2 dx_1 dx_2 dx_3 = \int_0^1 \int_0^{1-x_3} (1-x_2-x_3)^2 x_2 dx_2 dx_3$$

$$\begin{aligned}
&= \int_0^1 \int_0^{1-x_3} \frac{1}{3} (1-x_2-x_3)^3 dx_2 dx_3 \\
&= \int_0^1 \frac{1}{12} (1-x_3)^4 dx_3 \\
&= \left[\frac{-1}{60} (1-x_3)^5 \right]_0^1 = \frac{1}{60}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 6x_2 x_3 dx_1 dx_2 dx_3 &= \int_0^1 \int_0^{1-x_3} 6(1-x_2-x_3)x_2 x_3 dx_2 dx_3 \\
&= \int_0^1 \int_0^{1-x_3} 3(1-x_2-x_3)^2 x_2 x_3 dx_2 dx_3 \\
&= \int_0^1 (1-x_3)^3 x_3 dx_3 \\
&= \int_0^1 \frac{1}{4} (1-x_3)^4 dx_3 \\
&= \left[-\frac{1}{20} (1-x_3)^5 \right]_0^1 = \frac{1}{20}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 18x_3 x_1 dx_1 dx_2 dx_3 &= \int_0^1 \int_0^{1-x_3} 9x_3 (1-x_2-x_3)^2 dx_2 dx_3 \\
&= \int_0^1 3x_3 (1-x_3)^3 dx_3 \\
&= \int_0^1 \frac{3}{4} (1-x_3)^4 dx_3 \\
&= \left[-\frac{3}{20} (1-x_3)^5 \right]_0^1 = \frac{3}{20}
\end{aligned}$$

We conclude that:

$$\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 2x_1 x_2 + 6x_2 x_3 + 18x_3 x_1 dx_1 dx_2 dx_3 = \frac{1}{60} + \frac{1}{20} + \frac{3}{20} = \frac{13}{60}$$

Exercise 6 Given the curve

$$\gamma : [0, \pi] \rightarrow \mathbb{R}^2, \quad t \mapsto (3 \cos(2t), \sin(2t))$$

and a function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x_1, x_2) \mapsto (x_1^2 + 81x_2^2)^{\frac{3}{2}},$$

compute the integrals

$$\int_{\Gamma} f \, d\ell, \quad \int_{\Gamma} \nabla f \, d\ell.$$

Solution 6 *The integral of the gradient along this simple closed regular curve is zero. With regard to the integral of the scalar field along the curve, we have*

$$\begin{aligned}\dot{\gamma}(t) &= (-6 \sin(2t), 2 \cos(2t)), \\ |\dot{\gamma}(t)| &= \sqrt{36 \sin^2(2t) + 4 \cos^2(2t)} = 2\sqrt{9 \sin^2(2t) + \cos^2(2t)}.\end{aligned}$$

We also see that

$$\begin{aligned}f(\gamma(t)) &= (9 \cos^2(2t) + 81 \sin^2(2t))^{\frac{3}{2}} = 9^{\frac{3}{2}} (\cos^2(2t) + 9 \sin^2(2t))^{\frac{3}{2}} \\ &= 27 (\cos^2(2t) + 9 \sin^2(2t))^{\frac{3}{2}},\end{aligned}$$

In combination,

$$\begin{aligned}\int_{\Gamma} f d\ell &= \int_0^{\pi} f(\gamma(t)) \cdot |\dot{\gamma}(t)| dt \\ &= \int_0^{\pi} 27 (\cos^2(2t) + 9 \sin^2(2t))^{\frac{3}{2}} \cdot 2\sqrt{\cos^2(2t) + 9 \sin^2(2t)} dt \\ &= 54 \int_0^{\pi} (\cos^2(2t) + 9 \sin^2(2t))^2 dt.\end{aligned}$$

Our life will be easier if we substitute $u = 2t$. We also use a trigonometric identity. Then

$$\begin{aligned}&\int_0^{\pi} (\cos^2(2t) + 9 \sin^2(2t))^2 dt \\ &= \frac{1}{2} \int_0^{2\pi} (\cos^2(u) + 9 \sin^2(u))^2 du \\ &= \frac{1}{2} \int_0^{2\pi} (1 - \sin^2(u) + 9 \sin^2(u))^2 du \\ &= \frac{1}{2} \int_0^{2\pi} (1 + 8 \sin^2(u))^2 du \\ &= \frac{1}{2} \int_0^{2\pi} 1 + 16 \sin^2(u) + 64 \sin^4(u) du \\ &= \int_0^{\pi} 1 + 16 \sin^2(u) + 64 \sin^4(u) du \\ &= \pi + 16 \int_0^{\pi} \sin^2(u) du + 64 \int_0^{\pi} \sin^4(u) du.\end{aligned}$$

We use the half-angle formula:

$$\sin(u)^2 = \frac{1}{2} (1 - \cos(2u)).$$

Thus

$$\begin{aligned}
\int_0^\pi \sin^2(u) \, du &= \frac{1}{2} \int_0^\pi 1 - \cos(2u) \, du \\
&= \frac{\pi}{2} - \int_0^\pi \frac{1}{2} \cos(2u) \, du \\
&= \frac{\pi}{2} - \int_0^\pi \frac{1}{4} (\sin(2u))' \, du \\
&= \frac{\pi}{2} - \frac{1}{4} [(\sin(2u))']_0^\pi = \frac{\pi}{2}.
\end{aligned}$$

Similarly, using the half-angle formula twice:

$$\begin{aligned}
\int_0^\pi \sin^4(u) \, du &= \frac{1}{4} \int_0^\pi (1 - \cos(2u))^2 \, du \\
&= \frac{1}{4} \int_0^\pi 1 - 2\cos(2u) + \cos(2u)^2 \, du \\
&= \frac{1}{4} \int_0^\pi 1 - 2\cos(2u) + \frac{1}{2}(1 - \cos(4u)) \, du \\
&= \frac{1}{4} \int_0^\pi \frac{3}{2} - 2\cos(2u) - \frac{1}{2}\cos(4u) \, du \\
&= \frac{1}{4} \int_0^\pi \frac{3}{2} \, du - \frac{1}{2} \int_0^\pi \cos(2u) \, du - \frac{1}{8} \int_0^\pi \cos(4u) \, du \\
&= \frac{3}{8}\pi.
\end{aligned}$$

The integrals of $\cos(2u)$ and $\cos(4u)$ from 0 to π vanish, by the the same argument as above. In total,

$$\int_\Gamma f d\ell = 54 \left(\pi + 16 \cdot \frac{\pi}{2} + 64 \cdot \frac{3}{8}\pi \right) = 54 (\pi + 8\pi + 24\pi) = 54 \cdot 33\pi = 1782\pi.$$