Analysis III - 203(d)

Winter Semester 2024

Session 5: October 10, 2024

Exercise 1 *Consider a curve u in one-dimensional space with*

$$u : [1,2] \to \mathbb{R}, \qquad t \mapsto t^2 + t$$

Verify that the curve is simple, differentiable, and regular. Compute the curve integral $\int_u f\ dl$, where

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \to 3x^3$$

is the scalar field.

Solution 1 *This is just a fancy way of describing integration by substitution from Analysis I. We first verify that the curve is simple, differentiable, and regular.*

- *The curve is simple if it does not intersect itself. This means that $u(t_1) \neq u(t_2)$ for $t_1 \neq t_2$. In this case, the curve $u(t) = t^2 + t$ is a strictly increasing function on the interval $[1, 2]$. We conclude that the curve is simple.*
- *We see that the curve is differentiable because its only component is differentiable.*
- *The curve is regular because the derivative $\dot{u}(t) = 2t + 1$ is not zero over the interval $(1, 2)$.*

The curve integral of f over Γ is given by

$$\int_u f\, dl = \int_1^2 f(u(t))\, |\dot{u}(t)|\, dt = \int_1^2 3(u(t))^3\, |2t+1|\, dt = \int_1^2 3(t^2+t)^3\, |2t+1|\, dt.$$

We have $|2t + 1| = 2t + 1$ over the interval $[1, 2]$. Hence

$$\int_u f\, dl = 3 \int_1^2 (t^2+t)^3 (2t+1)\, dt.$$

This integral can be computed using standard methods of integration. One straight-forward but technical solution is to just expand the polynomial that we integrate. However, a simpler method uses substitution:

$$\int_u f\, dl = \frac{3}{4} \int_1^2 4(t^2+t)^3(2t+1)\, dt = \frac{3}{4} \int_1^2 \partial_t (t^2+t)^4\, dt = \frac{3}{4} (t^2+t)^4 |_{t=1}^{t=2} = \frac{3}{4} \left(6^4 - 2^4\right). \quad (1)$$

This simplifies to

$$\int_u f\, dl = \frac{3}{4} \cdot 2^4 \left(3^4 - 1\right) = \frac{3}{4} \cdot 16 \cdot 80 = 3 \cdot 320 = 960. \quad (2)$$

Exercise 2 (vector analysis in 1D) *Let $\Omega \subseteq (a, b)$ be an open interval in one-dimensional space.*

- *Explain why there cannot be a simple closed continuous curve in Ω.*
- *When $\Omega = (-10, 10)$, compute the integral of the scalar field*

$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

along the curves

$$\begin{aligned} \gamma_1 &: [0,1] \to \Omega, \quad t \mapsto (2t-1), \\ \gamma_2 &: [-1,1] \to \Omega, \quad t \mapsto (t), \\ \gamma_3 &: [0,1] \to \Omega, \quad t \mapsto (1-2t), \\ \gamma_4 &: [0,1] \to \Omega, \quad t \mapsto (-1+2t^5), \end{aligned}$$

Compute the tangent vectors $\dot{\gamma}(t)$.

- *Compute the integral of the vector field*

$$F(x) = \left(x e^{x^2} \right) \tag{3}$$

along the curve γ_4. Find a potential for this vector field, and write down the general form of all potentials.

Solution 2 *This is content of Analysis 1 but repackaged in the manner of vector analysis.*

- *It is visually clear that any closed continuous curve in Ω would have to intersect itself. Formally, one can use the intermediate value theorem.*
- *Before we compute all the line integrals, we note that the curves γ_1, γ_3, and γ_4 map the interval $[0,1]$ to the interval $[-1,1]$. Hence, we can solve the corresponding line integrals by a change of variables in the line integral for γ_2. Furthermore, we note that γ_3 is the reparameterization of γ_1 in the opposite direction. Hence, we conclude from Exercise 5 on Exercise Sheet 4, that the corresponding line integrals must be equal. Let us now start the computations with γ_2:*

$$\int_{\gamma_2} f \, dl = \int_{-1}^{1} f(t) \, dt = \int_{-1}^{1} \frac{t}{\sqrt{1+t^2}} \, dt = 0, \tag{4}$$

since the integrand is odd around $t = 0$. For γ_1, we have

$$\int_{\gamma_1} f \, dl = \int_0^1 f(\gamma_1(t)) |\dot{\gamma}_1(t)| \, dt = 2 \int_0^1 f(2t-1) \, dt = \int_{-1}^{1} f(s) \, ds = 0 = \int_{\gamma_2} f \, dl = 0. \tag{5}$$

As already pointed out previously, we must have $\int_{\gamma_3} f\, dl = \int_{\gamma_1} f\, dl = 0$*. Finally, we find for* γ_4 *that* $\dot\gamma_4(t) = 10t^4 \geq 0$ *for* $t \in [0,1]$ *and thus*

$$\int_{\gamma_4} f\, dl = \int_0^1 f(\gamma_4(t))\dot\gamma_4(t)\, dt = \int_{-1}^1 f(u)\, du = \int_{\gamma_2} f\, dl = 0, \tag{6}$$

where we have used the substitution $u = \gamma_4(t) = -1 + 2t^5$ *with* $du = \dot\gamma_4(t)\, dt$*.*

- *The general form of a potential for* F *is* $f(x) = \frac{1}{2}e^{x^2} + C$*, where* C *is an arbitrary constant. We set* $C = -1/2$ *such that* $f(0) = 0$*. For the curve integral of* F *along* γ_4*, we thus find that*

$$\int_{\gamma_4} F dl = f(\gamma_4(1)) - f(\gamma_4(0)) = f(1) - f(-1) = \frac{1}{2}(e - e^{(-1)^2}) = 0. \tag{7}$$

Exercise 3 *We review notions of potentials and conservative vector fields. Let* $\Omega \subseteq \mathbb{R}^n$ *be open. Suppose we have a vector field* $F = (F_1, \ldots, F_n) \in C^1(\Omega, \mathbb{R}^n)$*. Recall that we have introduced the condition*

$$\partial_i F_j = \partial_j F_i, \qquad 1 \leq i, j \leq n. \tag{8}$$

- *Suppose that* $n = 2$*. Show that* F *satisfies* (8) *if and only if it is curl-free:* $\operatorname{curl} F = 0$*.*
- *Suppose that* $n = 3$*. Show that* F *satisfies* (8) *if and only if it is curl-free:* $\operatorname{curl} F = 0$*.*
- *Suppose that* $n = 1$*. Show that* F *satisfies* (8)*.*
- *Suppose that* F *admits a potential* $f \in C^1(\Omega, \mathbb{R})$*, so that* $\nabla f = F$*. Show that if* $\gamma : [a, b] \to \Omega$ *is a simple regular curve, then*

$$\int_\gamma F\, dl = f(\gamma(b)) - f(\gamma(a)). \tag{9}$$

Show that if γ *is closed, then*

$$\int_\gamma F\, dl = 0. \tag{10}$$

Solution 3

- *We recall that the curl of a vector field* $F = (F_1, F_2)$ *is given by*

$$\operatorname{curl} F = (\partial_1 F_2 - \partial_2 F_1)\,. \tag{11}$$

Hence, it is clear that $\operatorname{curl} F = 0$ *if and only if* F *satisfies* (8)*.*

- *We recall that the curl of a vector field* $F = (F_1, F_2, F_3)$ *is given by*

$$\operatorname{curl} F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}. \tag{12}$$

Hence, it is clear that $\operatorname{curl} F = 0$ *if and only if* F *satisfies* (8). *Also, recall Exercise 6 from Exercise Sheet 3. On the one hand, we noticed there that the curl of a vector field is given by the off-diagonal entries of the anti-symmetric part of the Jacobian matrix. On the other hand, condition* (8) *is equivalent to the statement that the Jacobian matrix of* F *is symmetric. We therefore conclude that the curl vanishes if and only if the Jacobian matrix is symmetric.*

- *For* $n = 1$, *we have* $F = (F_1)$ *and the condition* (8) *is trivially satisfied.*

- *We have* $\nabla f = F$. *Hence, we can write the line integral as*

$$\int_\gamma F \, dl = \int_a^b F(\gamma(t)) \cdot \dot{\gamma}(t) \, dt \tag{13}$$

$$= \int_a^b \nabla f(\gamma(t)) \cdot \dot{\gamma}(t) \, dt = \int_a^b \frac{d}{dt} f(\gamma(t)) \, dt = f(\gamma(b)) - f(\gamma(a)). \tag{14}$$

If γ *is closed, then* $\gamma(a) = \gamma(b)$ *and we find that the line integral vanishes.*

Exercise 4 *We introduce the following curves:*

$$\gamma : [0,1] \to \mathbb{R}^3, \qquad t \mapsto \left(3, t^2, 4t\right),$$
$$\delta : [1, \infty) \to \mathbb{R}^2, \qquad t \mapsto \left(5, e^{-t}\right)$$

For each curve

- *compute the tangent vector*
- *compute the speed of the curve*
- *find the unit tangent vector*
- *for* δ, *find the unit normal along the curve that is the 90 degree clockwise rotation of unit tangent*
- *argue why it is a regular curve*
- *and compute the length of the curve.*

Solution 4

- *We compute the tangent vectors:*

$$\dot{\gamma}(t) = \begin{pmatrix} 0 \\ 2t \\ 4 \end{pmatrix}, \quad \dot{\delta}(t) = \begin{pmatrix} 0 \\ -e^{-t} \end{pmatrix}. \tag{15}$$

- *The speed of the curve is given by the norm of the tangent vector:*

$$|\dot{\gamma}(t)| = \sqrt{4t^2 + 16} = 2\sqrt{t^2 + 4}, \quad |\dot{\delta}(t)| = e^{-t}. \tag{16}$$

- *The unit tangent vector is given by*

$$\hat{t}_\gamma(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = \frac{1}{\sqrt{t^2 + 4}} \begin{pmatrix} 0 \\ t \\ 2 \end{pmatrix}, \tag{17}$$

$$\hat{t}_\delta(t) = \frac{\dot{\delta}(t)}{|\dot{\delta}(t)|} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{18}$$

  Note that the unit tangent vector of δ is constant!

- *The unit normal vector is given by the 90 degree clockwise rotation of the unit tangent vector. The corresponding rotation matrix is*

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{19}$$

  Hence, we find

$$\hat{n}_\delta(t) = R\hat{t}_\delta(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{20}$$

- *The curves are regular because their tangent vectors never vanish.*

- *The length of the curves is given by the integral of the speed: For γ, we find*

$$\int_0^1 |\dot{\gamma}(t)| dt = 2\int_0^1 \sqrt{t^2 + 4} dt = 4\int_0^1 \sqrt{(t/2)^2 + 1} dt \tag{21}$$

$$= 8\int_0^{1/2} \sqrt{u^2 + 1} du = 4(\sqrt{2} + \sinh^{-1}(1/2)), \tag{22}$$

$$\tag{23}$$

  where we have used the substitution $u = t/2$ with $du = dt/2$. The last integral can be found with trigonometric substitution.

  For δ, we find

$$\int_1^\infty |\dot{\delta}(t)| dt = \int_1^\infty e^{-t} dt = e^{-1}. \tag{24}$$

  Note that the curve is parametrized for $t \in [1, \infty)$, but still has a finite length!

Exercise 5 *We consider the vector field*

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto \left(x^3, y^3\right)$$

We want to find a potential over the domain $\Omega = \mathbb{R}^2$. Fix a constant of integration at $(0,0)$ and define a potential via the integral of the vector field F along a simple regular curve going from $(0,0)$ to (x,y).

Solution 5 *The most simple among the simple regular curves from $(0,0)$ to (x,y) is the straight line:*

$$\gamma : [0,1] \to \mathbb{R}^2, \qquad t \mapsto (tx, ty) .$$

We compute that $\dot{\gamma}(t) = (x, y)$. We fix some constant of integration $f(0,0) = C$ for our yet-to-be-found potential $f \in C^1(\mathbb{R}^2, \mathbb{R})$. For any $(x,y) \in \mathbb{R}^2$, we now compute

$$f(x,y) - f(0,0) = \int_0^1 \left(t^3x^3, t^3y^3\right) \cdot (x,y)\ dt = \left(x^4 + y^4\right) \int_0^1 t^3\ dt = \frac{1}{4}\left(x^4 + y^4\right) . \tag{25}$$

Therefore,

$$f(x,y) = \frac{1}{4}\left(x^4 + y^4\right) + C. \tag{26}$$

Indeed, one easily verifies that $\nabla f = F$.

Exercise 6 *The closed curve*

$$\gamma(t) = (\sin(t)(1 + 0.5\sin(2t)), \cos(t)(1 + 0.5\sin(2t)))$$

encircles a domain Ω in counterclockwise direction. Find the tangent $\dot{\gamma}(t)$, the unit tangent $\tau(t)$ and the outward pointing unit normal $\vec{n}(t)$. Only simplify as much as reasonable.

Solution 6 *We calculate:*

$$\dot{\gamma}(t) = (\cos(t)(1 + 0.5\sin(2t)) + \sin(t)\cos(2t), -\sin(t)(1 + 0.5\sin(2t)) + \cos(t)\cos(2t))$$

With that:

$$\begin{aligned}
|\dot{\gamma}(t)|^2 &= \cos(t)^2(1 + 0.5\sin(2t))^2 + \sin(t)^2\cos(2t)^2 + 2\cos(t)(1 + 0.5\sin(2t))\sin(t)\cos(2t) \\
&\quad + \sin(t)^2(1 + 0.5\sin(2t))^2 + \cos(t)^2\cos(2t)^2 - 2\sin(t)(1 + 0.5\sin(2t))\cos(t)\cos(2t) \\
&= (1 + 0.5\sin(2t))^2 + \cos(2t)^2 + 2\cos(t)(1 + 0.5\sin(2t))\sin(t)\cos(2t) \\
&\quad - 2\sin(t)(1 + 0.5\sin(2t))\cos(t)\cos(2t) \\
&= (1 + 0.5\sin(2t))^2 + \cos(2t)^2.
\end{aligned}$$

Hence

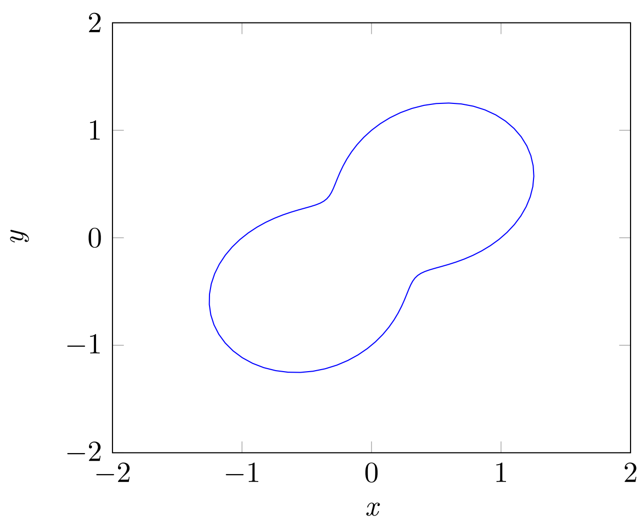
$$|\dot{\gamma}(t)| = \sqrt{(1 + 0.5\sin(2t))^2 + \cos(2t)^2}.$$

We want to compute the tangent vector:

$$\tau(t) = \left(\frac{\cos(t)(1 + 0.5\sin(2t)) + \sin(t)\cos(2t)}{\sqrt{(1 + 0.5\sin(2t))^2 + \cos(2t)^2}}, \frac{-\sin(t)(1 + 0.5\sin(2t)) + \cos(t)\cos(2t)}{\sqrt{(1 + 0.5\sin(2t))^2 + \cos(2t)^2}} \right)$$

Accordingly, the normal vector is:

$$\vec{n}(t) = \left(\frac{-\sin(t)(1 + 0.5\sin(2t)) + \cos(t)\cos(2t)}{\sqrt{(1 + 0.5\sin(2t))^2 + \cos(2t)^2}}, -\frac{\cos(t)(1 + 0.5\sin(2t)) + \sin(t)\cos(2t)}{\sqrt{(1 + 0.5\sin(2t))^2 + \cos(2t)^2}} \right)$$



Exercise 7 *We work over the quadratic domain*

$$\Omega := \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; -1 < x_1 < 1,\ -1 < x_2 < 1 \right\}.$$

Compute the integral $\iint_\Omega \mathrm{div}\vec{F}\, dx_1 dx_2$*, where*

$$\vec{F}(x_1, x_2) = \left(\sin(x_1)x_2, \left(x_1^2 + x_2\right)^5 \right)$$

Solution 7 *We make use of the Divergence theorem to express the volume integral as a curve integral.*

$$\int\int_\Omega \nabla \cdot \vec{F}\, dx_1\, dx_2 = \int_{\partial\Omega} \vec{F} \cdot \vec{n}\, d\ell,$$

$$
\begin{aligned}
&= \int_{-1}^{1} \vec{F}(x_1,-1)\cdot\begin{pmatrix}0\\-1\end{pmatrix}\, dx_1 + \int_{-1}^{1} \vec{F}(x_1,1)\cdot\begin{pmatrix}0\\1\end{pmatrix}\, dx_1,\\
&+ \int_{-1}^{1} \vec{F}(-1,x_2)\cdot\begin{pmatrix}-1\\0\end{pmatrix}\, dx_2 + \int_{-1}^{1} \vec{F}(1,x_2)\cdot\begin{pmatrix}1\\0\end{pmatrix}\, dx_2, s\\
&= \int_{-1}^{1} (1-x_1^2)^5\, dx_1 + \int_{-1}^{1} (x_1^2+1)^5\, dx_1 + \int_{-1}^{1} \sin(1)x_2\, dx_2 + \int_{-1}^{1} -\sin(-1)x_2\, dx_2,
\end{aligned}
$$

Since x_2 is an odd function, so the last two integrals are 0. To evaluate the first two integrals, we can either compute the quintic powers manually, which is a lot of work, or we utilize the binomial theorem:

$$
(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^{n-k}y^k
$$

With that, we can simplify the calculation as follows:

$$
\begin{aligned}
\int\int_{\Omega} \nabla\cdot\vec{F}\, dx_1\, dx_2 &= \int_{\partial\Omega} \vec{F}\cdot\vec{n}\, d\ell,\\
&= \int_{-1}^{1} (1-x_1^2)^5\, dx_1 + \int_{-1}^{1} (x_1^2+1)^5\, dx_1\\
&= \sum_{k=0}^{5}\binom{5}{k}(-1)^k \int_{-1}^{1} (x)^{2k}\, dx_1 + \sum_{k=0}^{5}\binom{5}{k}\int_{-1}^{1} (x)^{2k}\, dx_1\\
&= \sum_{k=0}^{5}\binom{5}{k}(-1)^k \left[\frac{x^{2k+1}}{2k+1}\right]_{-1}^{1} + \sum_{k=0}^{5}\binom{5}{k}\left[\frac{x^{2k+1}}{2k+1}\right]_{-1}^{1}\\
&= \sum_{k=0}^{5}\binom{5}{k}\Big((-1)^k+1\Big)\left[\frac{x^{2k+1}}{2k+1}\right]_{-1}^{1}
\end{aligned}
$$

For $k=1,3,5$, we get $\big((-1)^k+1\big)=0$. So we only need the terms with $k=0,2,4$. Thus

$$
\begin{aligned}
&\sum_{k=0}^{5}\binom{5}{k}\Big((-1)^k+1\Big)\left[\frac{x^{2k+1}}{2k+1}\right]_{-1}^{1}\\
&= 2\binom{5}{0}\left[\frac{x^{0+1}}{0+1}\right]_{-1}^{1} + 2\binom{5}{2}\left[\frac{x^{4+1}}{4+1}\right]_{-1}^{1} + 2\binom{5}{4}\left[\frac{x^{8+1}}{8+1}\right]_{-1}^{1}\\
&= 2\binom{5}{0}\left[\frac{x^{1}}{1}\right]_{-1}^{1} + 2\binom{5}{2}\left[\frac{x^{5}}{5}\right]_{-1}^{1} + 2\binom{5}{4}\left[\frac{x^{9}}{9}\right]_{-1}^{1}\\
&= 4\binom{5}{0} + \frac{4}{5}\binom{5}{2} + \frac{4}{9}\binom{5}{4}\\
&= \frac{128}{9}.
\end{aligned}
$$