Analysis III - 203(d)

Winter Semester 2024

Session 4: October 3, 2024

Exercise 1 *Compute the line integral of the vector field $\vec{F}$ along the curve γ, where*

$$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto (x_2, 0), \qquad \gamma : [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos(t), \sin(t))$$

Compute the line integral of the vector field $\vec{G}$ along the curve δ, where

$$\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto (2, 3, -1), \qquad \delta : [0, 4] \to \mathbb{R}^2, \quad t \mapsto (t^2, \cos(t), e^t)$$

Solution 1

$$\begin{aligned}
\int_\Gamma \vec{F}(x_1, x_2) d\ell &= \int_0^{2\pi} (\vec{F} \circ \gamma)(t) \cdot \dot{\gamma}(t) dt \\
&= \int_0^{2\pi} \begin{pmatrix} \sin t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} dt \\
&= \int_0^{2\pi} -\sin^2(t) dt \\
&= \int_0^{2\pi} -\frac{1}{2} + \frac{1}{2}\cos(2t) dt = \left[-\frac{1}{2}t + \frac{1}{4}\sin(2t) \right]_0^{2\pi} = -\pi
\end{aligned}$$

$$\begin{aligned}
\int_\Gamma \vec{G}(x_1, x_2, x_3) d\ell &= \int_0^4 (\vec{G} \circ \delta)(t) \cdot \dot{\delta}(t) dt \\
&= \int_0^4 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ -\sin(t) \\ e^t \end{pmatrix} dt \\
&= \int_0^4 4t - 3\sin(t) - e^t dt \\
&= \left[2t^2 + 3\cos(t) - e^t \right]_0^4 \\
&= 32 + 3\cos 4 - e^4 - (3 - 1) = 30 + 3\cos 4 - e^4
\end{aligned}$$

Exercise 2 *Compute the line integral of the vector field $\vec{F}$ along the curve γ, where*

$$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto (e^{x_1 x_2} x_2, e^{x_1 x_2} x_1)$$

and

$$\gamma : [0, 1] \to \mathbb{R}^2, \quad t \mapsto \left(\arctan(\cos(\pi t)^2 - \sin(\pi t)^2), 1 + \sqrt[2]{1 + \arctan(t)} \right)$$

Solution 2 *The vector field is the gradient of $e^{x_1 x_2}$, and so we can use the formula for vector fields that admit a potential. We find that*

$$\int_\Gamma \vec{F} d\ell = \int_\Gamma \nabla f d\ell = f(\gamma(1)) - f(\gamma(0))$$

Now we only need to compute the function values of f. Explicitly,

$$\begin{aligned}\gamma(0) &= \left(\arctan(\cos(\pi 0)^2 - \sin(\pi 0)^2), 1 + \sqrt[2]{1 + \arctan(0)}\right) \\ &= \left(\arctan(1), 1 + \sqrt[2]{1}\right) = (\pi/4, 2)\end{aligned}$$

$$\begin{aligned}\gamma(1) &= \left(\arctan(\cos(\pi 1)^2 - \sin(\pi 1)^2), 1 + \sqrt[2]{1 + \arctan(1)}\right) \\ &= \left(\arctan(1), 1 + \sqrt[2]{1 + \pi/4}\right) = \left(\pi/4, 1 + \sqrt[2]{1 + \pi/4}\right)\end{aligned}$$

We then compute

$$f(\gamma(1)) - f(\gamma(0)) = \exp\left(\frac{\pi}{4}(1 + \sqrt[2]{1 + \pi/4})\right) - \exp(\pi/2).$$

Exercise 3 *What is the length of the graph of the function $g(x) = x^{\frac{3}{2}}$ over the interval $[0, 2]$? Simplify as much as reasonable.*

Solution 3 *We can express the graph as a curve*

$$\gamma : [0, 2] \to \mathbb{R}^2, \quad t \mapsto (t, t^{\frac{3}{2}})$$

As mentioned in the lecture, the integral of 1 along that curve gives the length. Hence we find:

$$\int_\Gamma 1 \, dl = \int_0^2 \left| \left(1, \frac{3}{2} t^{\frac{1}{2}}\right) \right| \, dt = \int_0^2 \sqrt{1 + \frac{9t}{4}} \, dt$$

We integrate the last expression. The integrand is a derivative, up to some numerical factor:

$$\begin{aligned}\int_0^2 \sqrt{1 + \frac{9}{4} t} \, dt &= \frac{2}{3} \cdot \frac{4}{9} \int_0^2 \frac{3}{2} \cdot \frac{9}{4} \left(1 + \frac{9}{4} t\right)^{\frac{1}{2}} \, dt \\ &= \frac{2}{3} \cdot \frac{4}{9} \int_0^2 \partial_t \left(1 + \frac{9}{4} t\right)^{\frac{3}{2}} \, dt \\ &= \frac{2}{3} \cdot \frac{4}{9} \left[\left(1 + \frac{9}{4} t\right)^{\frac{3}{2}} \right]_0^2 \\ &= \frac{2}{3} \cdot \frac{4}{9} \left(\left(1 + \frac{9}{2}\right)^{\frac{3}{2}} - 1 \right)\end{aligned}$$

$$= \frac{8}{27}\left(\left(\frac{11}{2}\right)^{\frac{3}{2}} - 1\right).$$

(You can compute the integral via Intégration par changement de variable, which is the same calculation but in a different formalism.)

Exercise 4 *Take a look at the functions*

$$f(x_1, x_2) = 1, \quad g(x_1, x_2) = x_2.$$

We have the following curves:

$$\begin{aligned}
\alpha &: [0,1] \to \mathbb{R}^2, \quad t \mapsto (t, t), \\
\beta &: [0,2] \to \mathbb{R}^2, \quad t \mapsto \begin{cases} (t, 0) & 0 \leq t < 1, \\ (0, t-1) & 1 \leq t \leq 2, \end{cases} \\
\gamma &: [0,1] \to \mathbb{R}^2, \quad t \mapsto (t^2, t).
\end{aligned}$$

Compute:

$$\int_A f \, d\ell, \quad \int_B f \, d\ell, \quad \int_A g \, d\ell, \quad \int_B g \, d\ell, \quad \int_\Gamma g \, d\ell.$$

Solution 4 *We compute these integrals as follows:*

$$\int_A f \, d\ell = \int_0^1 \sqrt{2} \, dt = \left[\sqrt{2}t\right]_0^1 = \sqrt{2}, \quad \int_B f \, d\ell = \int_0^2 \, dt = [t]_0^2 = 2$$

$$\int_A g \, d\ell = \int_0^1 t\sqrt{2} \, dt = \left[\frac{\sqrt{2}}{2}t^2\right]_0^1 = \frac{\sqrt{2}}{2}$$

$$\int_B g \, d\ell = \int_0^1 (0) \, dt + \int_1^2 (t-1) \, dt = \int_1^2 (t-1) \, dt = \left[\frac{1}{2}(t-1)^2\right]_1^2 = \frac{1}{2}$$

For the interal of g over the third curve, we proceed as follows:

$$\begin{aligned}
\int_\Gamma g \, d\ell &= \int_0^1 t\sqrt{4t^2 + 1} \, dt \\
&= \int_0^1 t\left(4t^2 + 1\right)^{\frac{1}{2}} \, dt \\
&= \frac{2}{3} \cdot \frac{1}{8} \cdot \int_0^1 \frac{3}{2} \cdot 8t\left(4t^2 + 1\right)^{\frac{1}{2}} \, dt \\
&= \frac{2}{3} \cdot \frac{1}{8} \cdot \int_0^1 \partial_t \left(4t^2 + 1\right)^{\frac{3}{2}} \, dt = \frac{2}{3} \cdot \frac{1}{8} \cdot \left(5^{\frac{3}{2}} - 1\right) = \frac{5^{\frac{3}{2}} - 1}{12}.
\end{aligned}$$

Note that the curves all have the same starting and end points, $(0,0)$ and $(1,1)$, but the curve integrals of the function may (generally) differ from curve to curve.

Exercise 5 *We want to verify that the curve integrals do not depend on the direction of the parameterization. Consider the curve parameterizations*

$$\gamma_+ : [0,1] \to \mathbb{R}^2, \quad t \mapsto (t, 1 + \frac{1}{2}t^2), \tag{1}$$

$$\gamma_- : [0,1] \to \mathbb{R}^2, \quad s \mapsto (1-s, 1 + \frac{1}{2}(1-s)^2), \tag{2}$$

- *Given the scalar field*

  $$f(x,y) = x(y - \frac{1}{2}x^2) = xy - \frac{1}{2}x^3,$$

  show that

  $$\int_{\gamma_+} f \, dl = \int_{\gamma_-} f \, dl \tag{3}$$

- *Compute the tangent and unit tangent vectors of each curve, and show that $\dot{\gamma}_+(t) = -\dot{\gamma}_-(1-t)$. Show that the curve integrals along γ_+ and γ_- of the vector field*

  $$F(x,y) = (-y, x) \tag{4}$$

  are the negative of each other.

Solution 5

- *We compute the first integral directly:*

  $$\begin{aligned}\int_{\gamma_+} f \, dl &= \int_0^1 f(t, 1 + \frac{1}{2}t^2)|\dot{\gamma}_+(t)|dt \\ &= \int_0^1 t\sqrt{1+t^2}dt \\ &= \frac{1}{2}\int_0^1 2t\sqrt{1+t^2}dt.\end{aligned}$$

  We substitute $u = 1 + t^2$, so that $du = 2dt$. Thus

  $$\frac{1}{2}\int_0^1 2t\sqrt{1+t^2}dt = \frac{1}{2}\int_1^2 \sqrt{u}\, du = \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2 = \frac{1}{3}\left(2\sqrt{2} - 1\right).$$

  The second integral can be computed directly as well, or with a change of variables. For the latter, we use

  $$\begin{aligned}\int_{\gamma_-} f \, dl &= \int_0^1 f(1-s, 1 + \frac{1}{2}(1-s)^2)|\dot{\gamma}_-(s)|ds \\ &= -\int_1^0 f(t, 1 + \frac{1}{2}t^2)|\dot{\gamma}_-(1-t)|dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 t\sqrt{1+(1-(1-t))^2}dt = \int_0^1 t\sqrt{1+t^2}dt \\
&= \int_{\gamma_+} f dl,
\end{aligned}
$$

where we used the substitution $t = 1-s$ with $dt = -ds$.

- *Given the two parameterizations as stated above,*

$$
\gamma_+ : [0,1] \to \mathbb{R}^2, \quad t \mapsto (t, 1+\frac{1}{2}t^2), \tag{5}
$$

$$
\gamma_- : [0,1] \to \mathbb{R}^2, \quad s \mapsto (1-s, 1+\frac{1}{2}(1-s)^2), \tag{6}
$$

We compute the tangent vectors:

$$
\dot\gamma_+(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \qquad \dot\gamma_-(s) = \begin{pmatrix} -1 \\ s-1 \end{pmatrix}.
$$

Their norms are

$$
\begin{aligned}
|\dot\gamma_+(t)| &= \sqrt{1+t^2}, \\
|\dot\gamma_-(s)| &= \sqrt{1+(s-1)^2}.
\end{aligned}
$$

It is intuitive that $\dot\gamma_+(t) = -\dot\gamma_-(1-t)$, as we check easily:

$$
\dot\gamma_-(1-t) = -\begin{pmatrix} -1 \\ 1-t-1 \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix} = \dot\gamma_+(t).
$$

Finally, for the curve integral we find again, using the change of variables $s = 1-t$, hence $ds = -dt$, the following identity:

$$
\begin{aligned}
\int_{\gamma_+} F\, dl &= \int_0^1 F(\gamma_+(t)) \cdot \dot\gamma_+(t)\, dt \\
&= -\int_0^1 F(\gamma_-(1-t)) \cdot \dot\gamma_-(1-t)\, dt \\
&= \int_1^0 F(\gamma_-(s)) \cdot \dot\gamma_-(s)\, ds.
\end{aligned}
$$

We switch the bounds of integration into normal order and find

$$
\int_1^0 F(\gamma_-(s)) \cdot \dot\gamma_-(s)\, ds = -\int_0^1 F(\gamma_-(s)) \cdot \dot\gamma_-(s)\, ds = -\int_{\gamma_-} F\, dl.
$$

Exercise 6 *Consider the open cube*

$$
\Omega = (0,1)^2 = \left\{ (x,y) \in \mathbb{R}^2 : 0 < x, y < 1 \right\}.
$$

- *Show that Ω is convex.*
- *Let $z \in \Omega$. Show that Ω is star-shaped with respect to z.*
- *Show that Ω is simply-connected.*

For the last part, show that any two continuous curves with the same endpoints are homotopic relative to endpoints.

Solution 6

Graphically, it is clear that each segment between points of Ω is also contained in Ω. But we prove this formally.
Suppose that $a = (a_1, a_2), b = (b_1, b_2) \in \Omega$ and $t \in [0, 1]$. We want to show that $ta + (1 - t)b \in \Omega$. We compute that

$$ta + (1 - t)b = (ta_1 + (1 - t)b_1, ta_2 + (1 - t)b_2) .$$

This point is in Ω if each coordinate is in the interval $(0, 1)$. We show that for the first coordinate, because the argument for the second coordinate is almost the same. Clearly,

$$ta_1 + (1 - t)b_1 > t \min(a_1, b_1) + (1 - t) \min(a_1, b_1) = \min(a_1, b_1) > 0,$$

$$ta_2 + (1 - t)b_2 < t \max(a_2, b_2) + (1 - t) \max(a_2, b_2) = \max(a_2, b_2) < 1.$$

We conclude that Ω is convex.

As already observed, the segment between any two points $z, a \in \Omega$ is already contained in Ω. If we fix any point $z \in \Omega$, then Ω is star-shaped. Note that we do not use any property except that Ω is convex. Indeed, every convex domain is also star-shaped (with respect to any point inside).

Given two continuous curves $\gamma_0, \gamma_1 : [a, b] \to \Omega$ with $\gamma_0(a) = \gamma_1(a) =: g_a$ and $\gamma_0(b) = \gamma_1(b) =: g_b$, we define

$$\gamma : [a, b] \times [0, 1] \to \mathbb{R}^2$$

by setting

$$\gamma(t, s) = (1 - s) \cdot \gamma_0(t) + s \cdot \gamma_1(t).$$

For any $t \in [a, b]$ we have $\gamma_0(t), \gamma_1(t) \in \Omega$, and since Ω is convex, we have $(1-s)\cdot\gamma_0(t)+s\cdot\gamma_1(t) \in \Omega$ for all $s \in [0, 1]$. Hence γ defines a function

$$\gamma : [a, b] \times [0, 1] \to \Omega$$

Furthermore, γ is obviously continuous in both variables.

We verify the remaing conditions. For all $t \in [a,b]$ we have

$$\gamma(t,0) = (1-0)\cdot\gamma_0(t) + 0\cdot\gamma_1(t) = \gamma_0(t) \tag{7}$$

and

$$\gamma(t,1) = (1-1)\cdot\gamma_0(t) + 1\cdot\gamma_1(t) = \gamma_1(t). \tag{8}$$

Additionally, for all $s \in [0,1]$, we have

$$\gamma(a,s) = (1-s)\cdot\gamma_0(a) + s\cdot\gamma_1(a) = (1-s)\cdot g_a + s\cdot g_a = g_a \tag{9}$$

and

$$\gamma(b,s) = (1-s)\cdot\gamma_0(b) + s\cdot\gamma_1(b) = (1-s)\cdot g_b + s\cdot g_b = g_b. \tag{10}$$

This shows that γ satisfies the required axioms.