Analysis III - 203(d)

Winter Semester 2024

Session 2: September 19, 2024

Exercise 1 Sketch the level sets of the functions

$$f(x_1, x_2) = x_1 + x_2, \quad g(x_1, x_2) = x_1^2 x_2$$

for the values 0, 1, and -1. Compute the gradients ∇f and ∇g .

Solution 1

$$\nabla f(x_1, x_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \nabla g(x_1, x_2) = \begin{pmatrix} 2x_1 x_2 \\ x_1^2 \end{pmatrix}$$

For the level sets of $f(x_1, x_2)$ consider:

$$x_1 + x_2 = C \implies x_2 = -x_1 + C$$

therefore the level sets are lines that intersect the axis x_2 at C with slope -1. One can use a similar approach for $g(x_1, x_2)$. Note that for C = 0:

$$x_1^2 x_2 = 0 \implies x_1 = 0 \text{ or } x_2 = 0$$

which gives that the level sets for C = 0 are the two axis x_1 and x_2 .

Exercise 2 Compute the curl and the divergence of the following vector field:

$$\vec{f}(x_1, x_2, x_3) = (x_1 x_2, x_3^2 x_1, e^{x_1 - x_2})$$

Solution 2

$$\nabla \times \vec{f}(x_1, x_2, x_3) = \left(-e^{x_1 - x_2} - 2x_1 x_3, -e^{x_1 - x_2}, x_3^2 - x_1 \right)$$
$$\nabla \cdot \vec{f}(x_1, x_2, x_3) = x_2$$

Exercise 3 Show that for every three-dimensional vector field $\vec{f} : \mathbb{R}^3 \to \mathbb{R}^3$ we have

$$\nabla \cdot \left(\nabla \times \vec{f} \right) = 0$$

Show that for every scalar field $g: \mathbb{R}^3 \to \mathbb{R}$ we have

$$\nabla \times (\nabla q) = 0$$

Consider the following vector field:

$$\vec{A}(x_1, x_2, x_3) := (\sin(x_2), \sin(x_3), \sin(x_1))$$

Is this vector field a gradient of some scalar field?

Solution 3 Let $\vec{f} : \mathbb{R}^3 \to \mathbb{R}^3$ then:

$$\nabla \cdot (\nabla \times \vec{f}(x_1, x_2, x_3)) = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \cdot (\partial_{x_2} f_3 - \partial_{x_3} f_2, \partial_{x_3} f_1 - \partial_{x_1} f_3, \partial_{x_1} f_2 - \partial_{x_2} f_1)$$

$$= \partial_{x_1 x_2} f_3 - \partial_{x_1 x_3} f_2 + \partial_{x_2 x_3} f_1 - \partial_{x_2 x_1} f_3 + \partial_{x_3 x_1} f_2 - \partial_{x_3 x_2} f_1$$

We can interchange the order of the partial derivatives, since $\vec{f} \in \mathbb{C}^2$ to arrive at:

$$\nabla \cdot (\nabla \times \vec{f}(x_1, x_2, x_3)) = \partial_{x_1 x_2} f_3 - \partial_{x_2 x_1} f_3 + \partial_{x_3 x_1} f_2 - \partial_{x_1 x_3} f_2 + \partial_{x_2 x_3} f_1 - \partial_{x_3 x_2} f_1$$

$$= \partial_{x_2 x_1} f_3 - \partial_{x_2 x_1} f_3 + \partial_{x_1 x_3} f_2 - \partial_{x_1 x_3} f_2 + \partial_{x_3 x_2} f_1 - \partial_{x_3 x_2} f_1 = 0$$

$$\implies \nabla \cdot (\nabla \times \vec{f}(x_1, x_2, x_3)) = 0$$

Let $g: \mathbb{R}^3 \to \mathbb{R}$ then:

$$\begin{aligned} \nabla \times \left(\nabla g(x_1, x_2, x_3) \right) &= \nabla \times \begin{pmatrix} \partial_{x_1} g \\ \partial_{x_2} g \\ \partial_{x_3} g \end{pmatrix} = \begin{pmatrix} \partial_{x_2 x_3} g - \partial_{x_3 x_2} g \\ \partial_{x_3 x_1} g - \partial_{x_1 x_3} g \\ \partial_{x_1 x_2} g - \partial_{x_2 x_1} g \end{pmatrix} = \begin{pmatrix} \partial_{x_2 x_3} g - \partial_{x_2 x_3} g \\ \partial_{x_1 x_3} g - \partial_{x_1 x_3} g \\ \partial_{x_1 x_2} g - \partial_{x_1 x_2} g \end{pmatrix} = \vec{0} \\ \implies \nabla \times \left(\nabla g(x_1, x_2, x_3) \right) = 0 \end{aligned}$$

$$\vec{\nabla} \times \vec{A}(x_1, x_2, x_3) := (-\cos(x_3), \cos(x_1), -\cos(x_2))$$

The curl of this vector field is not zero. If it were a gradient, then it would be zero. So it cannot be a gradient of any function.

Exercise 4 Given scalar fields $f, g : \mathbb{R}^3 \to \mathbb{R}$ and vector fields $\vec{A}, \vec{B} : \mathbb{R}^3 \to \mathbb{R}^3$, show that

$$\begin{split} \nabla \left(f \cdot g \right) &= f \cdot \nabla g + g \cdot \nabla f, \\ \nabla \cdot \left(f \cdot \vec{A} \right) &= (\nabla f) \cdot \vec{A} + f(\nabla \cdot \vec{A}), \\ \nabla \cdot \left(\vec{A} \times \vec{B} \right) &= \left(\nabla \times \vec{A} \right) \cdot \vec{B} - \vec{A} \cdot \left(\nabla \times \vec{B} \right), \\ \Delta (fg) &= f \Delta g + 2 \nabla (f) \cdot \nabla (g) + g \Delta f. \end{split}$$

Suppose that we also have two real numbers $\alpha, \beta \in \mathbb{R}$. Show that

$$\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g,$$
$$\nabla \cdot (\alpha \vec{A} + \beta \vec{B}) = \alpha \nabla \cdot \vec{A} + \beta \nabla \cdot \vec{B},$$
$$\nabla \times (\alpha \vec{A} + \beta \vec{B}) = \alpha \nabla \times \vec{A} + \beta \nabla \times \vec{B}.$$

Solution 4 For the first identity, we use the product rule:

$$\nabla (f \cdot g) = (\partial_1 (f \cdot g), \dots, \partial_n (f \cdot g))$$
$$= (\partial_1 f \cdot g + f \cdot \partial_1 g, \dots, \partial_n f \cdot g + f \cdot \partial_n g)$$

$$= (\partial_1 f \cdot g, \dots, \partial_n f \cdot g) + (f \cdot \partial_1 g, \dots, f \cdot \partial_n g)$$

$$= g (\partial_1 f, \dots, \partial_n f) + f (\partial_1 g, \dots, \partial_n g)$$

$$= f \cdot \nabla g + g \cdot \nabla f.$$

The second identity, again, can be proved using the product rule:

$$\nabla \cdot \left(f \cdot \vec{A} \right) = \partial_1(f \cdot A_1) + \dots + \partial_n(f \cdot A_n)$$

$$= \partial_1(f \cdot A_1) + \dots + \partial_n(f \cdot A_n)$$

$$= (\partial_1 f) \cdot A_1) + \dots + (\partial_n f) \cdot A_n + f \cdot \partial_1(A_1) + \dots + f \cdot \partial_n(A_n)$$

$$= \nabla(f) \cdot \vec{A} + f \cdot \nabla \cdot \vec{A},$$

As for the third identity:

$$\nabla \cdot \left(\vec{A} \times \vec{B} \right) = \nabla \cdot \left(A_2 B_3 - A_3 B_2, -A_1 B_3 + A_3 B_1, A_1 B_2 - A_2 B_1 \right)$$

$$= \partial_1 \left(A_2 B_3 - A_3 B_2 \right) + \partial_2 \left(-A_1 B_3 + A_3 B_1 \right) + \partial_3 \left(A_1 B_2 - A_2 B_1 \right)$$

$$= +\partial_1 \left(A_2 B_3 \right) - \partial_1 \left(A_3 B_2 \right)$$

$$- \partial_2 \left(A_1 B_3 \right) + \partial_2 \left(A_3 B_1 \right)$$

$$+ \partial_3 \left(A_1 B_2 \right) - \partial_3 \left(A_1 B_2 \right)$$

$$= +\partial_1 \left(A_2 \right) B_3 + A_2 \partial_1 \left(B_3 \right) - \partial_1 \left(A_3 \right) B_2 - A_3 \partial_1 \left(B_2 \right)$$

$$- \partial_2 \left(A_1 \right) B_3 - A_1 \partial_2 \left(B_3 \right) + \partial_2 \left(A_3 \right) B_1 + A_3 \partial_2 \left(B_1 \right)$$

$$+ \partial_3 \left(A_1 \right) B_2 + A_1 \partial_3 \left(B_2 \right) - \partial_3 \left(A_2 \right) B_1 - A_2 \partial_3 \left(B_1 \right)$$

$$= +\partial_2 \left(A_3 \right) B_1 - \partial_3 \left(A_2 \right) B_1 - \partial_1 \left(A_3 \right) B_2 + \partial_3 \left(A_1 \right) B_2 + \partial_1 \left(A_2 \right) B_3 - \partial_2 \left(A_1 \right) B_3$$

$$+ A_1 \partial_3 \left(B_2 \right) - A_1 \partial_2 \left(B_3 \right) - A_2 \partial_3 \left(B_1 \right) + A_2 \partial_1 \left(B_3 \right) + A_3 \partial_2 \left(B_1 \right) - A_3 \partial_1 \left(B_2 \right)$$

$$= \left(\nabla \times \vec{A} \right) \cdot \vec{B} - \left(\nabla \times \vec{B} \right) \cdot \vec{A}.$$

We can prove the fourth identity again via direct computation, which is perfectly valid. We can also use some of the identities already shown above to reduce the effort:

$$\begin{split} &\Delta(fg) = \nabla \cdot (\nabla(fg)) \\ &= \nabla \cdot (g\nabla(f) + f\nabla(g)) \\ &= \partial_1(g\nabla(f)_1 + f\nabla(g)_1) + \dots + \partial_n(g\nabla(f)_n + f\nabla(g)_n) \\ &= \partial_1(g\nabla(f)_1) + \partial_1(f\nabla(g)_1) + \dots + \partial_n(g\nabla(f)_n) + \partial_n(f\nabla(g)_n) \\ &= \partial_1(g\nabla(f)_1) + \dots + \partial_n(g\nabla(f)_n) + \partial_1(f\nabla(g)_1) + \dots + \partial_n(f\nabla(g)_n) \\ &= \nabla \cdot (g\nabla(f)) + \nabla \cdot (f\nabla(g)) \\ &= \nabla(g) \cdot \nabla(f) + g \cdot \nabla \cdot (\nabla f) + \nabla(f) \cdot \nabla(g) + f \cdot \nabla \cdot (\nabla g) = 2\nabla(f) \cdot \nabla(g) + f\Delta g + g\Delta f. \end{split}$$

The last three identities describe that gradient, divergence and curl are linear, which is a concept that you have already seen in linear algebra. We verify by direct computation:

$$\nabla(\alpha f + \beta g) = (\alpha \partial_1 f_1 + \beta \partial_1 g_1, \cdots, \alpha \partial_n f_n + \beta \partial_n g_n)$$

$$= (\alpha \partial_1 f_1, \dots, \alpha \partial_n f_n) + (\beta \partial_1 g_1, \dots, \beta \partial_n g_n)$$

= $\alpha (\partial_1 f_1, \dots, \partial_n f_n) + \beta (\partial_1 g_1, \dots, \partial_n g_n)$.

Similarly,

$$\nabla \cdot (\alpha \vec{A} + \beta \vec{B}) = \nabla \cdot (\alpha A_1 + \beta B_1, \cdots, \alpha A_n + \beta B_n)$$

$$= \alpha \partial_1 A_1 + \beta \partial_1 B_1 + \cdots + \alpha \partial_n A_n + \beta \partial_n B_n$$

$$= \alpha \partial_1 A_1 + \cdots + \alpha \partial_n A_n + \beta \partial_1 B_1 + \cdots + \beta \partial_n B_n$$

$$= \alpha \left(\partial_1 A_1 + \cdots + \partial_n A_n \right) + \beta \left(\partial_1 B_1 + \cdots + \partial_n B_n \right)$$

$$= \alpha \nabla \cdot \vec{A} + \beta \nabla \vec{B}.$$

Lastly,

$$\nabla \times (\alpha \vec{A} + \beta \vec{B})$$

$$= \nabla \times (\alpha A_1 + \beta B_1, \alpha A_2 + \beta B_2, \alpha A_3 + \beta B_3)$$

$$= (\alpha \partial_2 A_3 + \beta \partial_2 B_3 - \alpha \partial_3 A_2 - \beta \partial_3 B_2, -\alpha \partial_1 A_3 - \beta \partial_1 B_3 + \alpha \partial_3 A_1 + \beta \partial_3 B_1, \alpha \partial_1 A_2 + \beta \partial_1 B_2 - \alpha \partial_2 A_1 - \beta \partial_2 B_1)$$

$$= (\alpha \partial_2 A_3 - \alpha \partial_3 A_2 + \beta \partial_2 B_3 - \beta \partial_3 B_2, -\alpha \partial_1 A_3 + \alpha \partial_3 A_1 - \beta \partial_1 B_3 + \beta \partial_3 B_1, \alpha \partial_1 A_2 - \alpha \partial_2 A_1 + \beta \partial_1 B_2 - \beta \partial_2 B_1)$$

$$= \alpha (\partial_2 A_3 - \partial_3 A_2, -\partial_1 A_3 + \partial_3 A_1, \partial_1 A_2 - \partial_2 A_1) + \beta (\partial_2 B_3 - \partial_3 B_2, -\partial_1 B_3 + \partial_3 B_1, \partial_1 B_2 - \partial_2 B_1)$$

$$= \alpha \nabla \times \vec{A} + \beta \nabla \times \vec{B}.$$

Exercise 5 Consider the scalar field

$$\Phi: \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto ||x||^{-1}$$

and compute its gradient and Laplacian. Show that Φ is harmonic if n = 3. When n = 3, then this is the graviational potential, and its gradient is the gravitational field.

Solution 5 We remember that

$$\nabla \|x\| = \frac{x}{\|x\|}.$$

Hence

$$\partial_i \Phi(x) = (-1) ||x||^{-2} \cdot \frac{x_i}{||x||} = -\frac{x_i}{||x||^3}.$$

That mans

$$\nabla \Phi = -\frac{x}{\|x\|^3}.$$

We compute also the second derivative in direction i:

$$\partial_i \partial_i \Phi(x) = -\frac{1}{\|x\|^3} - x_i \cdot (-3) \|x\|^{-4} \frac{x_i}{\|x\|} = -\frac{1}{\|x\|^3} + 3 \frac{x_i^2}{\|x\|^5}.$$

We easily see that the Laplacian equals

$$\Delta\Phi(x) = \frac{n-3}{\|x\|^3}$$

In particular, the scalar field Φ is "harmonic" if n=3: We call a scalar field harmonic if its Laplacian equals zero.