Analysis III - 203(d)

Winter Semester 2024

Session 1: September 12, 2024

Exercise 1 *Compute the norms (that is, lengths), the* 3 *scalar products and* 6 *vector products of the following vectors. Find the volume of the parallelepiped spanned by these vectors.*

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \vec{c} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Solution 1

$$\vec{a} = \sqrt{6}, \quad \vec{b} = \sqrt{5}, \quad \vec{c} = \sqrt{11}$$
$$\vec{a} \cdot \vec{b} = 0, \quad \vec{a} \cdot \vec{c} = 0, \quad \vec{b} \cdot \vec{c} = 1,$$
$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} = \begin{pmatrix} -1 \\ -2 \\ -5 \end{pmatrix}, \quad \vec{a} \times \vec{c} = -\vec{c} \times \vec{a} = \begin{pmatrix} 7 \\ -4 \\ -1 \end{pmatrix}, \quad \vec{b} \times \vec{c} = -\vec{c} \times \vec{b} = \begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix},$$

The volume of the parallelepiped spanned by the three vectors is 18.

Exercise 2 *Consider the function*

$$f(x, y) = \sin\left(2\pi x^2 + 2\pi y^2\right)$$

Find the first derivatives in x and y. Sketch the level sets of f for the values 0 *and* 1.

Solution 2 *We easily compute*

$$\partial_x f(x, y) = \cos\left(2\pi x^2 + 2\pi y^2\right) \cdot 4\pi \cdot x, \quad \partial_y f(x, y) = \cos\left(2\pi x^2 + 2\pi y^2\right) \cdot 4\pi \cdot y.$$

The level sets are concentric circles centered at zero. One approach to understanding works as follows. First, we rewrite

$$f(x, y) = \sin\left(2\pi(x^2 + y^2)\right)$$

We remember that

$$\sin(2\pi t) = 0 \textit{ for the positive values } t = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

and that

$$\sin(2\pi t) = 1 \textit{ for the positive values } t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \ldots.$$

Hence, whenever $x^2 + y^2$ hits one these values listed above, then we are on the level of 0 or 1. Specifically, $f(x, y) = 0$ if (x, y) lies on a circle centered at zero and with radius

$$0, \sqrt{\frac{1}{2}}, \sqrt{1}, \sqrt{\frac{3}{2}}, \sqrt{2}, \ldots,$$

Similarly, $f(x, y) = 1$ if (x, y) lies on a circle centered at zero and with radius

$$0, \sqrt{\frac{1}{4}}, \sqrt{\frac{5}{4}}, \sqrt{\frac{9}{2}}, \sqrt{\frac{13}{4}}, \ldots.$$

Exercise 3 *Sketch the following vector field:*

$$f(x, y) = \begin{pmatrix} \sin(x) \\ sin(y) \end{pmatrix}$$

Solution 3 *Please do this in the exercise session.*

Exercise 4 *Compute the following integrals:*

$$\int_0^1 \int_0^1 x^2 e^y \, dxdy, \quad \int_0^1 x^2 \cos(x) \, dx.$$

Solution 4 *For the first integral, we can either split up the integral into two factors, or we compute one after the other. We find*

$$\int_0^1 \int_0^1 x^2 e^y \, dxdy = \int_0^1 x^2 \, dx \cdot \int_0^1 e^y \, dy = \frac{1}{3} \cdot (e - 1).$$

For the second integral, we use integration by parts.

$$\begin{aligned}
\int_0^1 x^2 \cos(x) \, dx &= x^2 \sin(x)|_0^1 - \int_0^1 2x \sin(x) \, dx \\
&= x^2 \sin(x)|_0^1 - \left(2x(-\cos(x))|_0^1 - \int_0^1 2(-\cos(x)) \, dx \right) \\
&= x^2 \sin(x)|_0^1 - 2x(-\cos(x))|_0^1 + \int_0^1 2(-\cos(x)) \, dx \\
&= x^2 \sin(x)|_0^1 - 2x(-\cos(x))|_0^1 - 2 \int_0^1 \cos(x) \, dx \\
&= x^2 \sin(x)|_0^1 - 2x(-\cos(x))|_0^1 - 2 \left(\sin(1) - \sin(0)\right) \\
&= \sin(1) - 2(-\cos(1)) - 2\sin(1) \\
&= 2\cos(1) - \sin(1).
\end{aligned}$$