October 17

Divergence theorem

As a specific example, consider the vector field

$$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3),$$

the source vector field. Using our parameterization of the sphere,

$$\oiint_S \vec{F} \cdot \vec{n} \, d\sigma = \int_0^{2\pi} \int_0^{\pi} \cos^2\theta \sin^3\varphi + \sin^2\theta \sin^3\varphi + \sin\varphi \cos^2\varphi \, d\varphi \, d\theta$$

outward pointing unit normal

$$= \int_0^{2\pi} \int_0^{\pi} \sin^3\varphi + \sin\varphi \cos^2\varphi \, d\varphi \, d\theta$$

$$= 2\pi \int_0^{\pi} \sin^3\varphi + \sin\varphi \cos^2\varphi \, d\varphi$$

$$= 2\pi \int_0^{\pi} \sin\varphi \underbrace{\left(\sin^2\varphi + \cos^2\varphi \right)}_{=1} d\varphi$$

$$= 2\pi \underbrace{\int_0^{\pi} \sin\varphi \, d\varphi}_{=2} = 4\pi$$

Let's get back to the divergence theorem

$$\oiint_S \vec{F} \cdot \vec{n} \, d\sigma = \iiint_V \operatorname{div} \vec{F} \, dx_1 dx_2 dx_3 = \iiint_V 3 \, dx_1 dx_2 dx_3$$

$$= 3 \cdot \left(\begin{array}{c} \text{volume of} \\ \text{3D unit ball} \end{array} \right)$$

This shows that the volume of the 3D unit ball equals

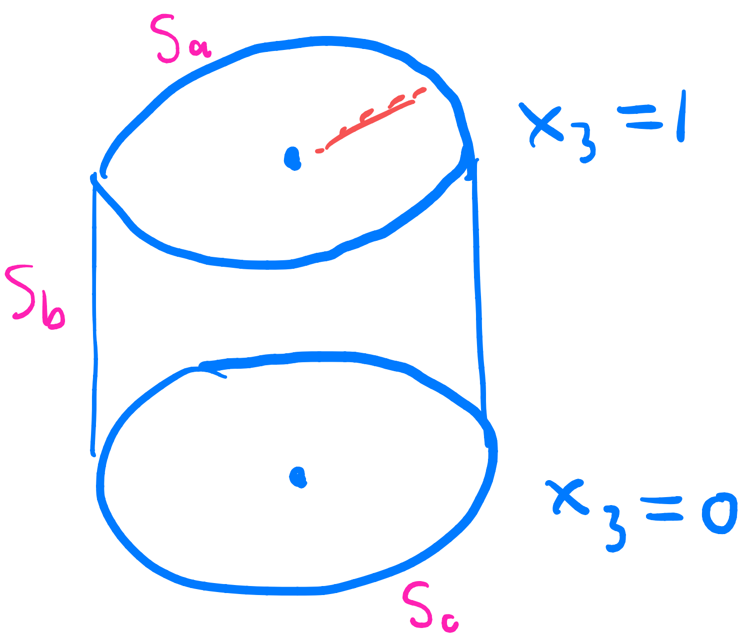
$$\frac{1}{3} \oiint \vec{F} \cdot \vec{n} \, d\sigma = \frac{4}{3} \pi$$

2. Example: Cylinder

The cylinder is the volume

$$V = \{ \vec{x} \in \mathbb{R}^3 \mid 0 < x_3 < 1, \quad x_1^2 + x_2^2 < 1$$

whose surface has three pieces.



$$S_a = \{ (x_1, x_2, 1) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1 \}$$

$$S_b = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1, \ 0 < x_3 < 1 \}$$

$$S_c = \{ (x_1, x_2, 0) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1 \}$$

For the divergence theorem:

$$\iiint_V \operatorname{div} \vec{F} \, dx_1 dx_2 dx_3 = \oiint_S \vec{F} \cdot \vec{n} \, d\sigma$$

$$= \iint_{S_a} \vec{F} \cdot \vec{n}_a \, d\sigma + \iint_{S_b} \vec{F} \cdot \vec{n}_b \, d\sigma + \iint_{S_c} \vec{F} \cdot \vec{n}_c \, d\sigma ,$$

we need to find the outward pointing unit normal and the parameterization that we use for the integral

We choose the following parameterizations

$$\Phi_a : \underbrace{\{ s^2 + t^2 < 1 \}}_{\Omega_a} \longrightarrow S_a , \qquad (s,t) \longmapsto (s,t,1)$$

$$\Phi_b : \underbrace{(0, 2\pi) \times (0,1)}_{\Omega_b} \to S_b, \quad (\theta, z) \mapsto (\cos\theta, \sin\theta, z)$$

$$\Phi_c : \underbrace{\{ s^2 + t^2 < 1 \}}_{\Omega_c} \to S_c, \quad (s, t) \mapsto (s, t, 0)$$

We compute the normals given by these parameterizations

$$\partial_s \Phi_a \times \partial_t \Phi_a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ pointing <u>outward</u> along top plate

$$\partial_s \Phi_c \times \partial_t \Phi_c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ pointing <u>inward</u> along bottom plate

$$\partial_\theta \Phi_b \times \partial_z \Phi_b = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$ pointing <u>outward</u> along shell

For illustration, we pick the following vector field

$$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x_1, x_2, x_3) = (x_2, x_1, x_3 e^{x_3})$$

We compute the boundary integrals

Top cap:
$$\iint_{S_a} \vec{F} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} d\sigma = \iint_{\Omega_a} 1 e^1 \, ds dt = \pi e$$

Bottom cap:
$$\iint_{S_c} \vec{F} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} d\sigma = \iint_{\Omega_c} 0 e^0 \, ds dt = 0$$

Cylinder shell:
$$\iint_{S_b} \vec{F} \cdot \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} d\sigma = \iint_{S_b} \begin{pmatrix} x_2 \\ x_1 \\ x_3 e^{x_3} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} d\sigma$$

$$= \int_0^1 \int_0^{2\pi} 2 \cos\theta \sin\theta \cdot 1 \, d\theta \, dz$$

$$= \int_0^1 \int_0^{2\pi} \partial_\theta \left(\sin^2\theta \right) d\theta \, dz$$

$$= \int_0^1 \sin^2\theta \Big|_{\theta=0}^{\theta=2\pi} dz \quad = \quad 0$$

In summary

$$\iiint_V \operatorname{div} \vec{F} \, dx_1 dx_2 dx_3 = \oiint_S \vec{F} \cdot \vec{n} \, d\sigma$$

$$= \underbrace{\iint_{S_a} \vec{F} \cdot \vec{n}_a}_{\pi e} + \underbrace{\iint_{S_b} \vec{F} \cdot \vec{n}_b}_{0} + \underbrace{\iint_{S_c} \vec{F} \cdot \vec{n}_c}_{0} = \pi e$$