# October 16

Surface integrals & Divergence theorem

# General Remark on Integrals

The notations $\int$, $\iint$, $\iiint$, $\oint$, $\oiint$ are conventions in mathematics / physics but do not carry particular meaning, it just a way to remember what the geometric object looks like

The notation $dx$, $dy$, $ds\,dt$, $dl$, $d\sigma$ is just notation. Historically, they used to signify "infinitissimal" quantities, but that formalism is no longer used. In that sense, the notation is "fossilized"

# Agenda

We have seen the divergence theorem over a two-dimensional domain $\Omega$ whose boundary is a curve $\partial\Omega$.

We generalize this to 3D volumes $V$ whose boundary is a surface $S$.

## Theory:

Let $V \subseteq \mathbb{R}^3$ be a domain. Let $S = \partial V$ be its boundary.

For any differentiable vector field

$$\vec{F} : \overline{V} \longrightarrow \mathbb{R}^3$$

we have

$$\iiint_V \operatorname{div} \vec{F} \, dx_1 dx_2 dx_3 = \oiint_S \vec{F} \cdot \vec{n} \, d\sigma$$

outward pointing unit normal

volume integral

closed surface integral

The physical idea is the same as in 2D

$$\iiint_V \operatorname{div} \vec{F} \, dx_1 dx_2 dx_3 = \oiint_S \vec{F} \cdot \vec{n} \, d\sigma$$

what is produced/consumed inside $V$

what flows out/in

## Practical challenge

- Find a parameterization of the surface $S = \partial V$
- Find the outward pointing unit normal
- Compute the integral

How does this interact with the parameterizations of $S$

Suppose we have a piecewise regular parameterization $\Phi : \Omega \to S$. Then

$$\oiint_S \vec{F} \cdot \vec{n} \, d\sigma = \iint_{\Omega} \vec{F}(\Phi(s,t)) \cdot \vec{n}(\Phi(s,t)) \, \| \partial_s \Phi(s,t) \times \partial_t \Phi(s,t) \| \, ds\,dt$$

Since $\partial_s \Phi$ and $\partial_t \Phi$ are tangential to $S$, we must have that $\partial_s \Phi \times \partial_t \Phi$ is normal to $S$. But we do not know whether that is an inward or outward normal.

<u>If</u> outward, then $\vec{n}(\Phi(s,t)) = \dfrac{\partial_s \Phi \times \partial_t \Phi}{\| \partial_s \Phi \times \partial_t \Phi \|}$

If inward, then $\vec{n}(\Phi(s,t)) = -\dfrac{\partial_s \Phi \times \partial_t \Phi}{\| \partial_s \Phi \times \partial_t \Phi \|}$

Hence, if $\partial_s \Phi \times \partial_t \Phi$ points outwards, then

$$\oiint_S \vec{F} \cdot \vec{n} \, d\sigma = \iint_\Omega \vec{F}(\Phi(s,t)) \cdot \frac{\partial_s \Phi \times \partial_t \Phi}{\| \partial_s \Phi \times \partial_t \Phi \|} \| \partial_s \Phi \times \partial_t \Phi \| \, ds dt$$

$$= \iint_\Omega \vec{F}(\Phi(s,t)) \cdot (\partial_s \Phi \times \partial_t \Phi) \, ds dt$$

Instead, if $\partial_s \Phi \times \partial_t \Phi$ points inwards, then

$$\oiint_S \vec{F} \cdot \vec{n} \, d\sigma = - \iint_\Omega \vec{F}(\Phi(s,t)) \cdot (\partial_s \Phi \times \partial_t \Phi) \, ds dt$$

# Examples

## 1. Sphere

We recall the sphere parameterization

$$\Phi : \underset{\theta}{(0, 2\pi)} \times \underset{\varphi}{(0, \pi)} \to \mathbb{R}^3,$$

$$(\theta, \varphi) \mapsto (\cos\theta \sin\varphi, \ \sin\theta \sin\varphi, \ \cos\varphi)$$

We recall

$$\partial_\theta \Phi = \begin{pmatrix} -\sin\theta \sin\varphi \\ \cos\theta \ \sin\varphi \\ 0 \end{pmatrix}, \qquad \partial_\varphi \Phi = \begin{pmatrix} \cos\theta \ \cos\varphi \\ \sin\theta \ \cos\varphi \\ -\sin \varphi \end{pmatrix}$$

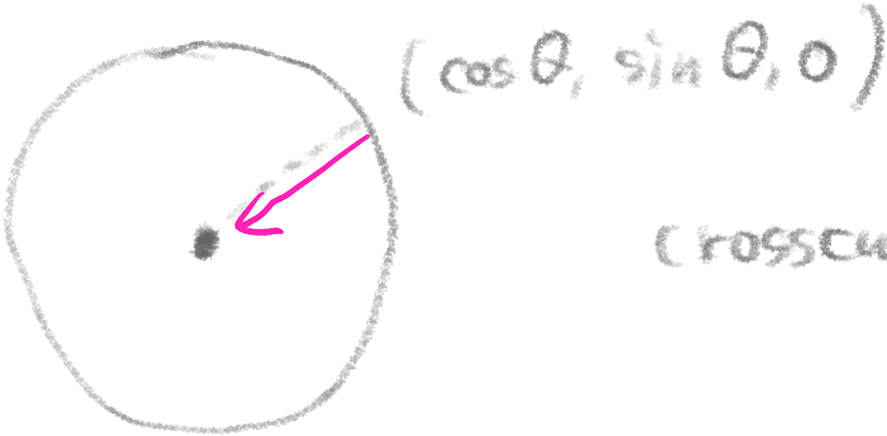
$$\partial_\theta \Phi \times \partial_\varphi \Phi = -\begin{pmatrix} \cos\theta \, \sin^2\varphi \\ \sin\theta \, \sin^2\varphi \\ \sin\varphi \, \cos\varphi \end{pmatrix}$$

Is it outward or inward pointing along $S$.?

Let's check at $(\theta, \varphi) = (\theta, \pi/2)$

$$(\partial_\theta \Phi \times \partial_\varphi \Phi)(\theta, \pi/2) = -\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

inward pointing!




crosscut at $x_3 = 0$

If $\partial_\theta \Phi \times \partial_\varphi \Phi$ points inwards, then

$-\partial_\theta \Phi \times \partial_\varphi \Phi$ points outwards

$$\iiint_V \operatorname{div} \vec{F}\, dx_1\, dx_2\, dx_3 = -\iint_\Omega \vec{F}(\Phi(s,t)) \cdot \left(\partial_\theta \Phi \times \partial_\varphi \Phi\right) d\theta\, d\varphi$$