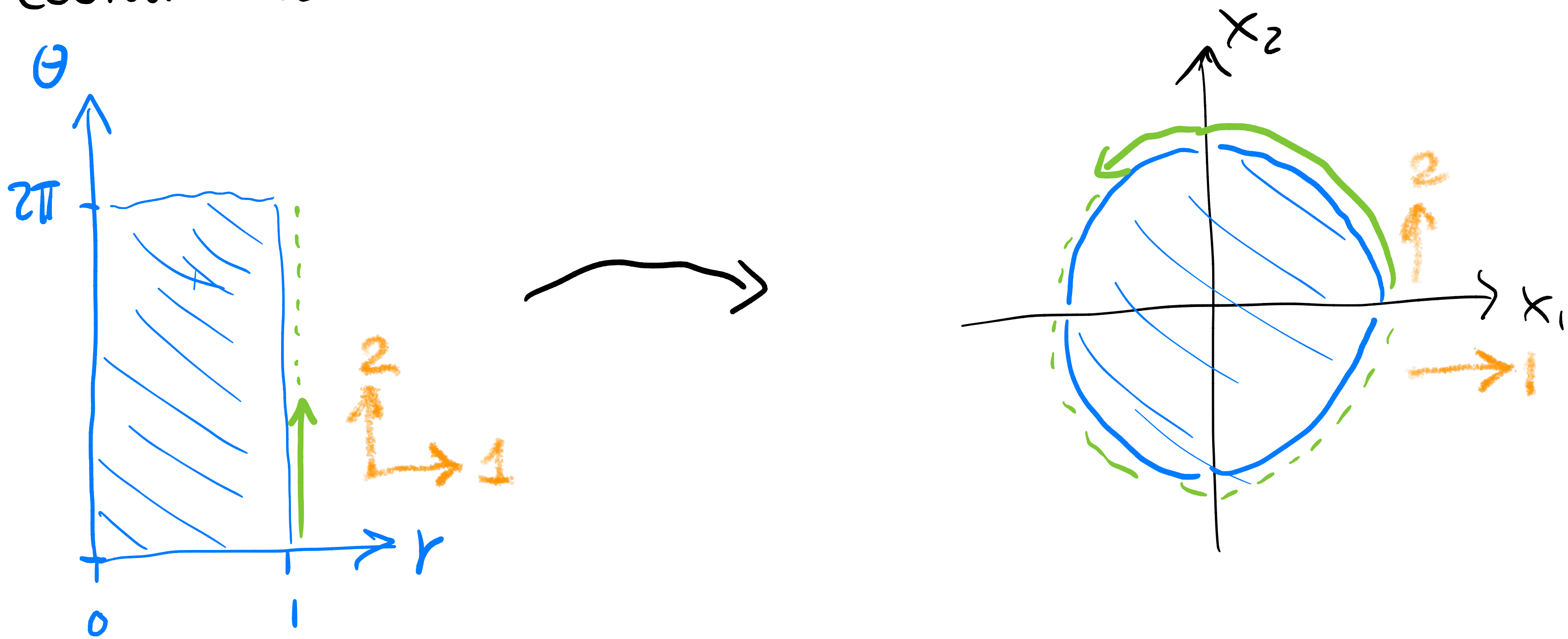
October 31

Stokes theorem

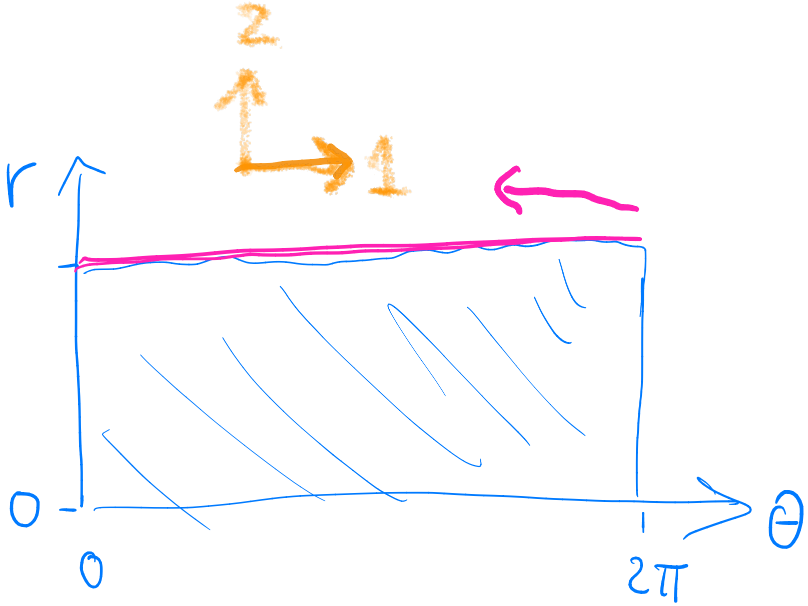
Moving on to the next example, suppose we have radial coordinates

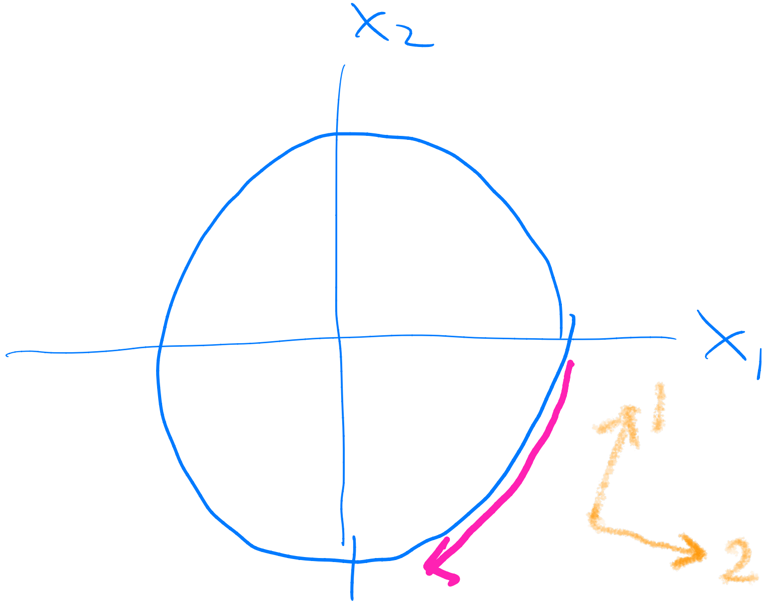


Here, $\Omega = (0,1) \times (0, 2\pi)$

The boundary $\partial\Omega$ is not mapped onto ∂S bijectively

We need some care when choosing the boundary parametrization



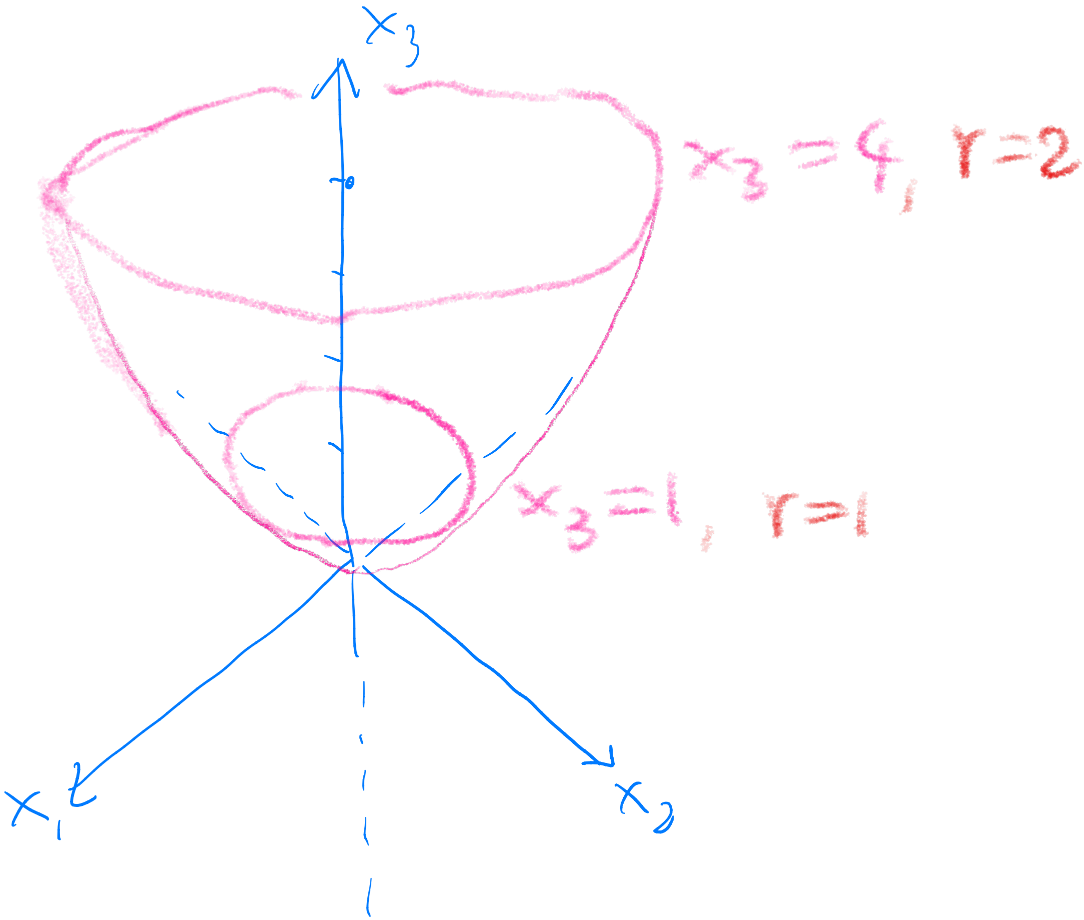


Switching the coordinates changes the orientation of the surface (direction of unit normal) and the orientation of the boundary (direction of the tangent)

We always walk CCW along $\partial\Omega$ and if some boundary part is mapped to ∂S, then this gives the orientation of the surface boundary.

Example : Bowl

$$S = \{ \vec{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = x_3 , \quad 0 < x_3 < 4 \}$$



At height r^2 we draw a circle of radius r.

How to parametrize?

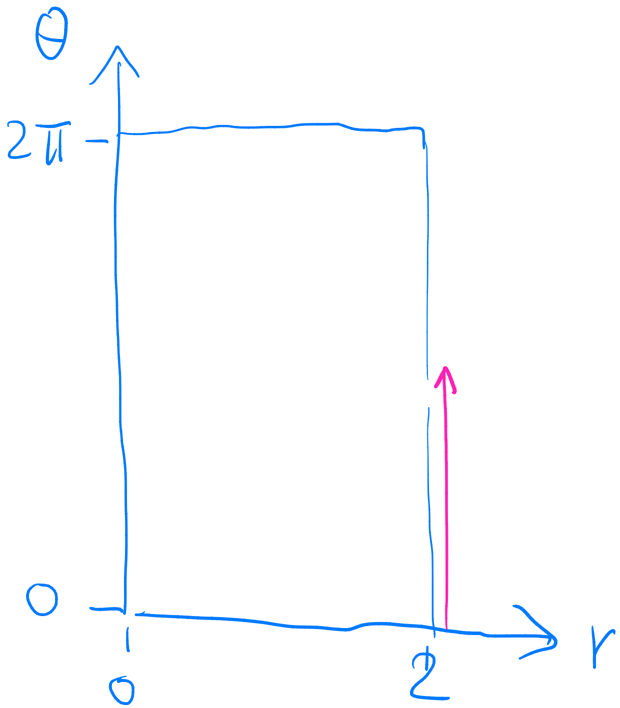
We parametrize S via

$$\Omega := (0,2) \times (0, 2\pi)$$

$$\Phi : \Omega \to \mathbb{R}^3, \quad (r, \theta) \longmapsto (r\cos\theta,\ r\sin\theta,\ r^2)$$

The boundary curve is parametrized by

$$\Phi \circ \gamma : [0, 2\pi) \longrightarrow \mathbb{R}^3, \quad \theta \longmapsto (2\cos\theta,\ 2\sin\theta,\ 4)$$



This is based on the curve

$$\gamma : [0, 2\pi) \to \mathbb{R}^2, \quad \theta \mapsto (2, \theta)$$

in the parameter domain.

We calculate

$$\partial_r \Phi(r,\theta) = (\cos\theta, \ \sin\theta, \ 2r)$$

$$\partial_\theta \Phi(r,\theta) = (-r\sin\theta, \ r\cos\theta, \ 0)$$

$$\partial_r \Phi(r,\theta) \times \partial_\theta \Phi(r,\theta) = (-2r^2\cos\theta, \ -2r^2\sin\theta, \ r)$$

We can use Stokes' theorem with an exmple vector field:

$$\vec{F}(x_1, x_2, x_3) = (x_2, \ x_3, \ x_1)$$

Hence,

$$\operatorname{curl} \vec{F}(x_1, x_2, x_3) = (-1, -1, -1)$$

$$\iint_S \operatorname{curl} \vec{F} \, d\sigma = \iint_\Omega \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2r^2 \cos\theta \\ -2r^2 \sin\theta \\ r \end{pmatrix} dr\, d\theta$$

$$= \int_0^2 \int_0^{2\pi} \cancel{r^2 (2\cos\theta + 2\sin\theta)} - r \, d\theta \, dr$$

$$= \int_0^2 \int_0^{2\pi} (-r) \, d\theta \, dr = -2\pi \int_0^2 r \, dr = -4\pi r$$

Let's compare with the boundary integral

$$\int_C \vec{F} \, dl = \int_0^{2\pi} F(\Phi \circ \gamma(t)) \cdot (\Phi \circ \gamma)'(t) \, dt$$

$$\Phi \circ \gamma = (2\cos\theta, 2\sin\theta, 4)$$

$$= \int_0^{2\pi} \begin{pmatrix} 2\sin\theta \\ 4 \\ 2\cos\theta \end{pmatrix} \cdot \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \\ 0 \end{pmatrix} d\theta = \int_0^{2\pi} -4\sin^2\theta + 8\cos\theta \quad d\theta$$

$$= \int_0^{2\pi} -4\sin^2\theta \; d\theta \underset{\text{Half-angle formula}}{=} -4\pi$$

That verifies Stokes' theorem in this example.

Yet another look at radial coordinates: