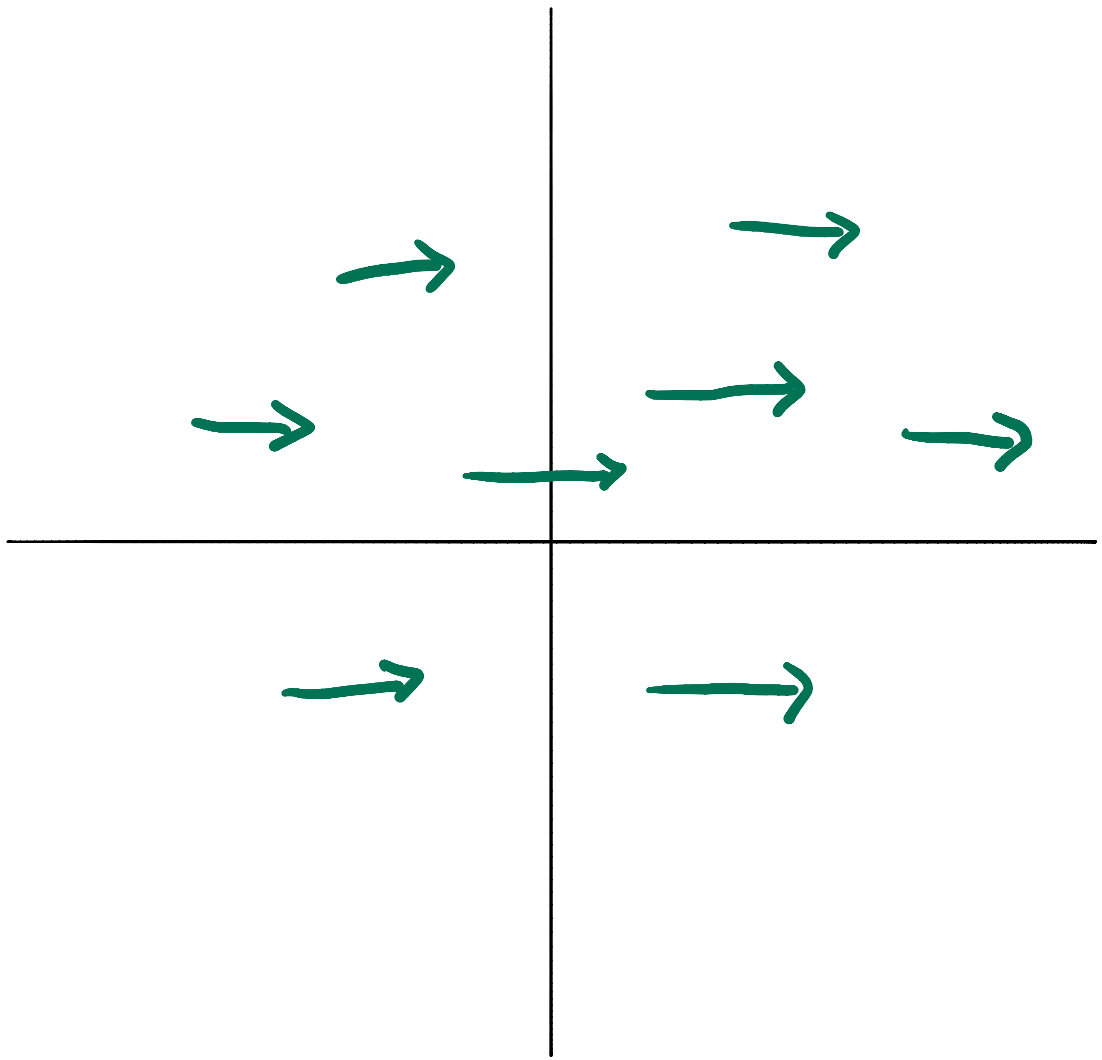
Interlude

Examples for

Gauss, Green, and potentials

Example: $F(x,y) = (1,0)$



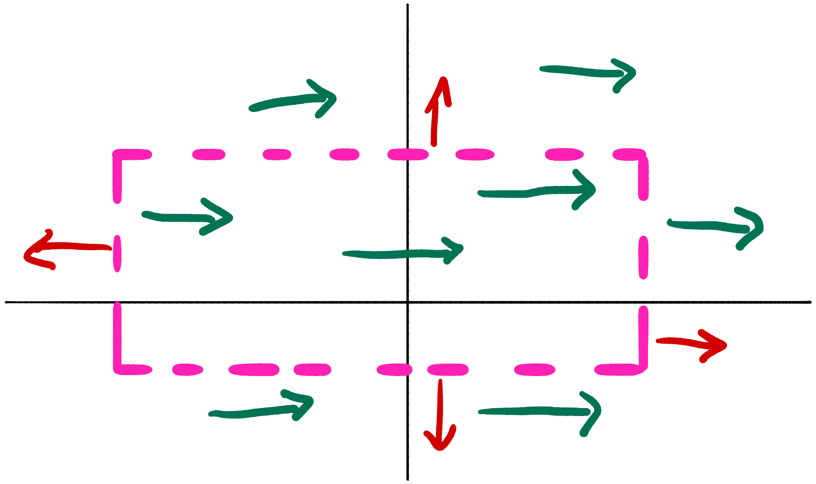
This vector field models transport with constant speed and direction

Obviously:

$$\operatorname{div} F = 0$$

$$\operatorname{curl} F = 0$$

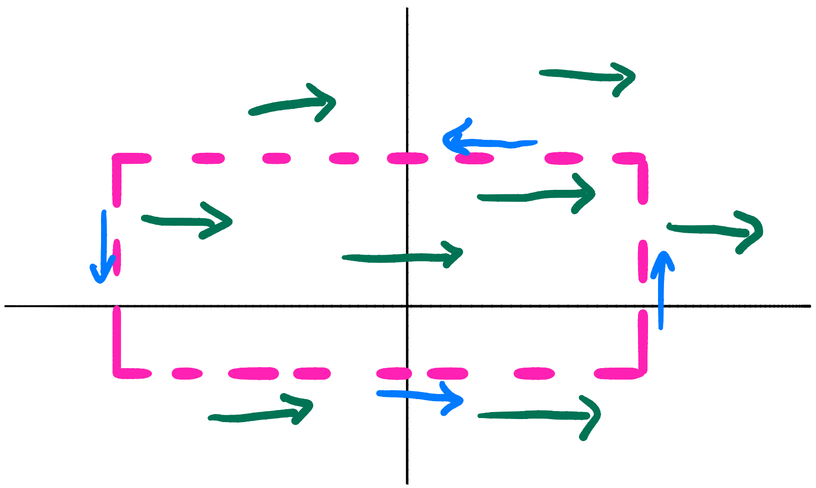
We verify the theorems of Gauss and Green at a rectangle



Gauss / divergence theorem

$$\iint_{\Omega} \operatorname{div} F(x,y)\, dx dy = 0$$

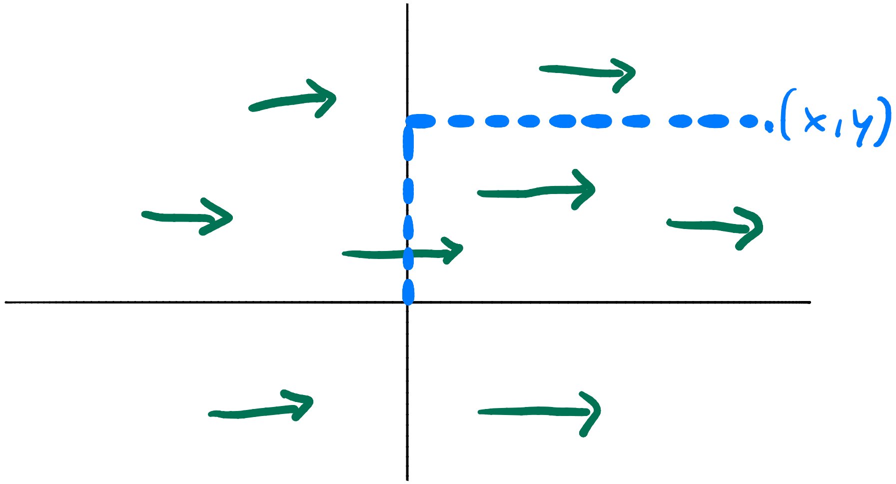
$$\oint_{\partial\Omega} F(x,y)\cdot\vec{n} = 0$$



Green's theorem

$$\iint_{\Omega} \operatorname{curl} F(x,y)\, dx dy = 0$$

$$\oint_{\partial\Omega} F\cdot\vec{\jmath} = 0$$



Over $\mathbb{R}^2$, the condition curl $F = 0$

$$\partial_x F_y - \partial_y F_x = 0$$

is enough to establish the existence of a potential $f \in C^1(\mathbb{R}^2, \mathbb{R})$. Let's find one

$$\nabla f = F$$

$\boxed{1.}$ We pick $z = (0,0)$ and set $f(0,0) = C$

$\boxed{2.}$ Given $y \in \mathbb{R}$, we integrate from $(0,0)$ to $(0,y)$

We choose the straight line segment $\gamma : [0,1] \to \mathbb{R}^2$, $t \mapsto (0, ty)$

$$f(0,y) - f(0,0) = \int_0^1 F(\gamma(t)) \cdot \dot{\gamma}(t)\, dt = \int_0^1 (1,0) \cdot (0,y)\, dt = 0$$

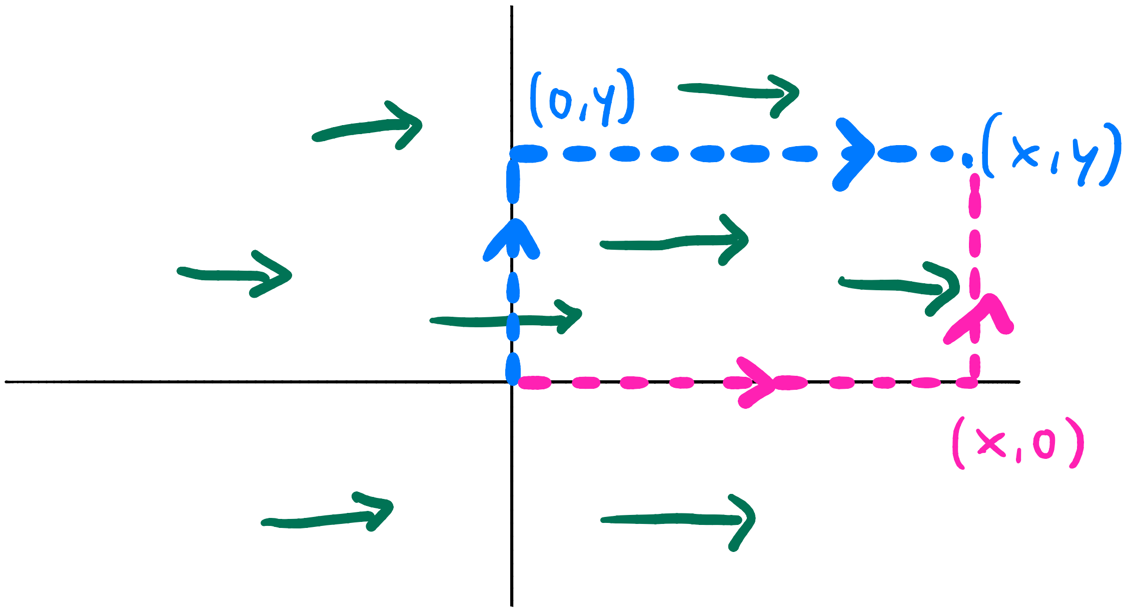
$$\Rightarrow \quad f(0,y) = f(0,0) + 0 = C$$

$\boxed{3}$ Given $(x,y) \in \mathbb{R}^2$, we integrate from $(0,y)$ to (x,y).

We use the straight line segment $\delta : [0,1] \to \mathbb{R}^2$, $t \mapsto (tx, y)$

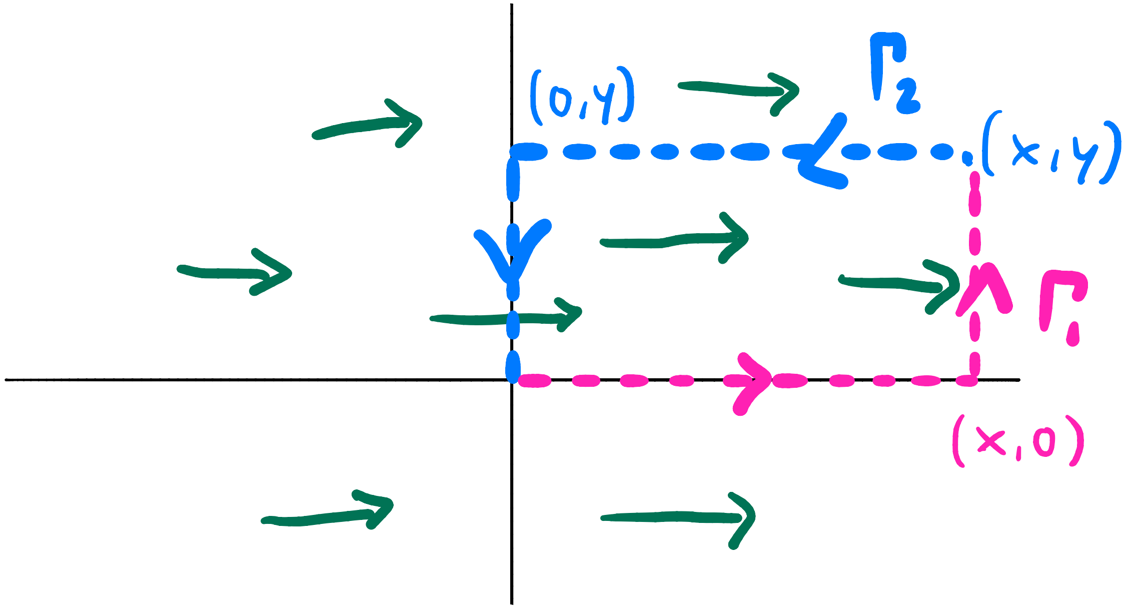
$$f(x,y) - f(0,y) = \int_0^1 F(\delta(t)) \cdot \dot{\delta}(t)\, dt = \int_0^1 (1,0) \cdot (x,0)\, dt = x$$

$$f(x,y) = f(0,y) + x = x + C$$



We have integrated along the straight line from (0,0) to (0,y) and then to (x,y)

We could have integrated instead from (0.0) to (x,0) and then (x,y). Because F has a potential, the result is the same.

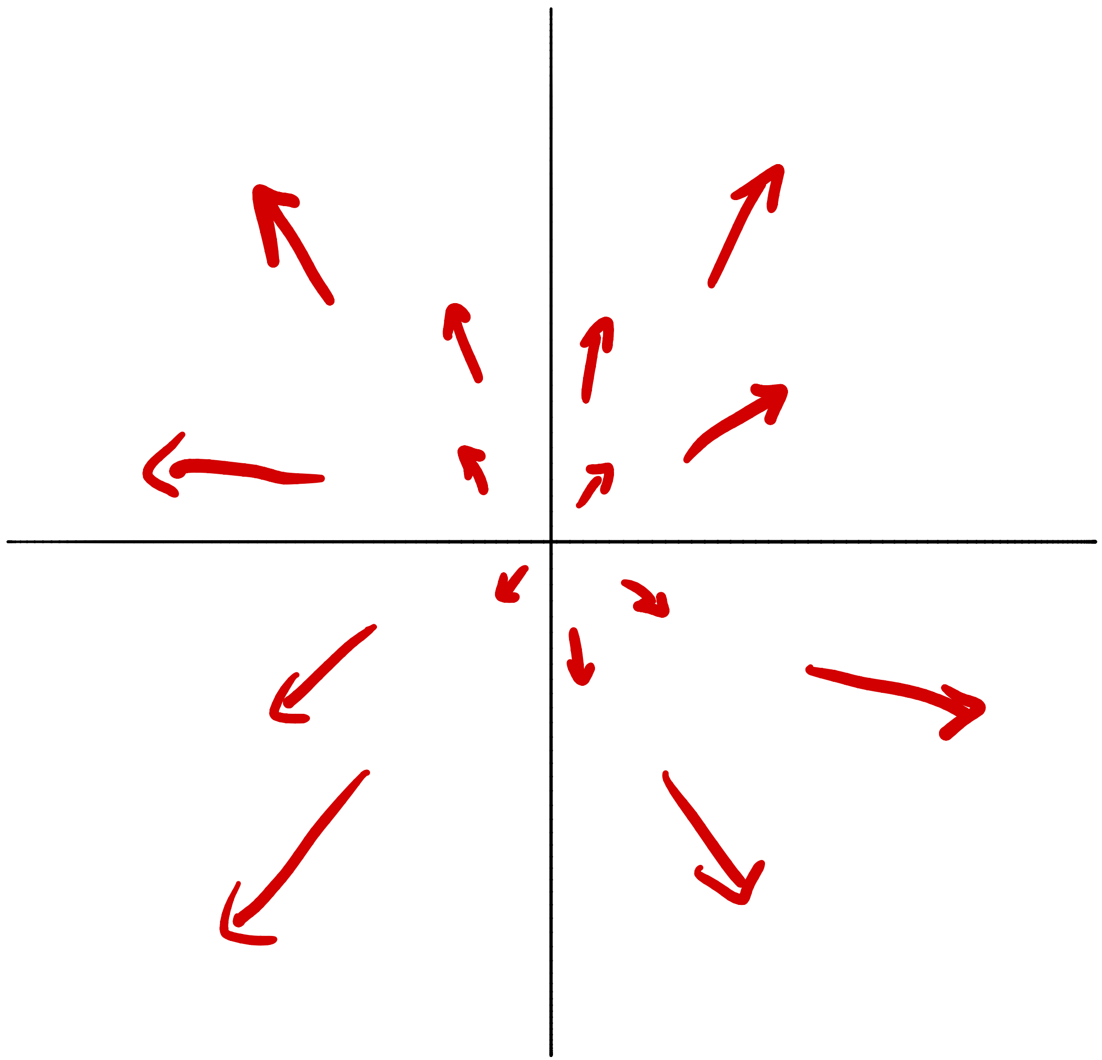


If we integrate from $(0,0)$ to $(x,0)$, then to (x,y), then to $(0,y)$, and back to $(0,0)$,

we have an integral of F along a closed piecewise regular (counterclockwise) curve.

$$\int_{\Gamma_1} F\,dl + \int_{\Gamma_2} F\,dl = \iint_{\Omega} \operatorname{curl} F\,dx\,dy = 0$$

Example: $F(x,y) = (x,y)$



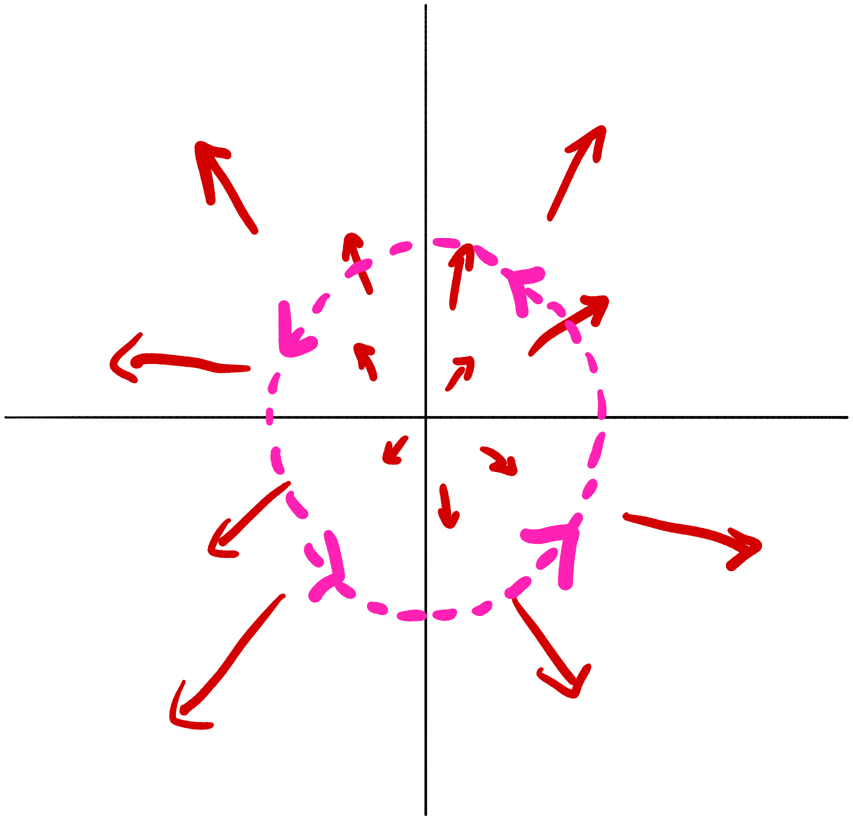
This vector field models flow from the origin

Obviously:

$$\operatorname{div} F = 2$$

$$\operatorname{curl} F = 0$$

We verify the theorems of Gauss and Green at a circular region



Consider the regular closed curve

$$\gamma : [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos t, \sin t)$$

Tangential vector

$$\dot\gamma(t) = (-\sin t, \cos t)$$

Unit tangent vector

$$T(t) = (-\sin t, \cos t)$$

Unit normal vector

$$n(t) = \frac{(\dot\gamma_2(t), -\dot\gamma_1(t))}{\|\dot\gamma(t)\|} = (\cos t, \sin t)$$

Verify Green's theorem

$$0 = \iint_{\Omega} \operatorname{curl} F \, dx\,dy = \oint_{\partial\Omega} F \, dl = \int_0^{2\pi} (\cos t, \sin t) \cdot (-\sin t, \cos t) \, dt = 0$$

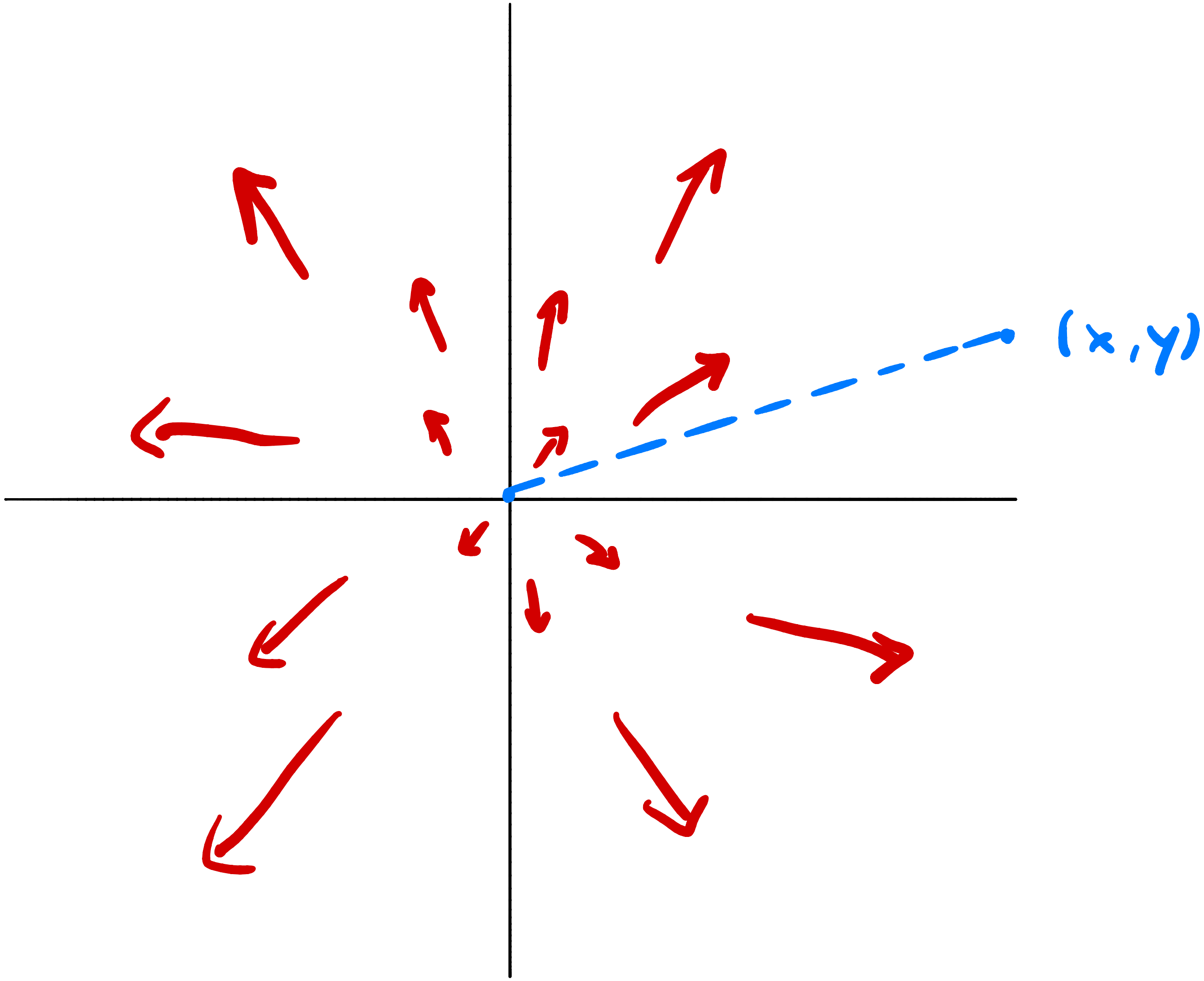
Verify divergence theorem

$$\iint_{\Omega} \operatorname{div} F \, dx\,dy = \iint_{\Omega} 2 \, dx\,dy = \operatorname{area}(\Omega) \cdot 2 = 2\pi$$

$$\oint_{\partial\Omega} F \cdot \vec{n} = \int_0^{2\pi} F(\gamma(t)) \cdot \vec{n} \, |\dot{\gamma}(t)| \, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t) \, dt = 2\pi$$

We construct a potential $f \in C^1(\mathbb{R}^2, \mathbb{R})$ for this vector field.

$\nabla f = F$



We construct a potential $f \in C^1(\mathbb{R}^2, \mathbb{R})$ for this vector field.

① We choose a pivot point where we start. For example

$$f(0,0) = C$$

② Given any $(x,y) \in \mathbb{R}^2$, we integrate along some curve from $(0,0)$ to (x,y). We choose the straight line

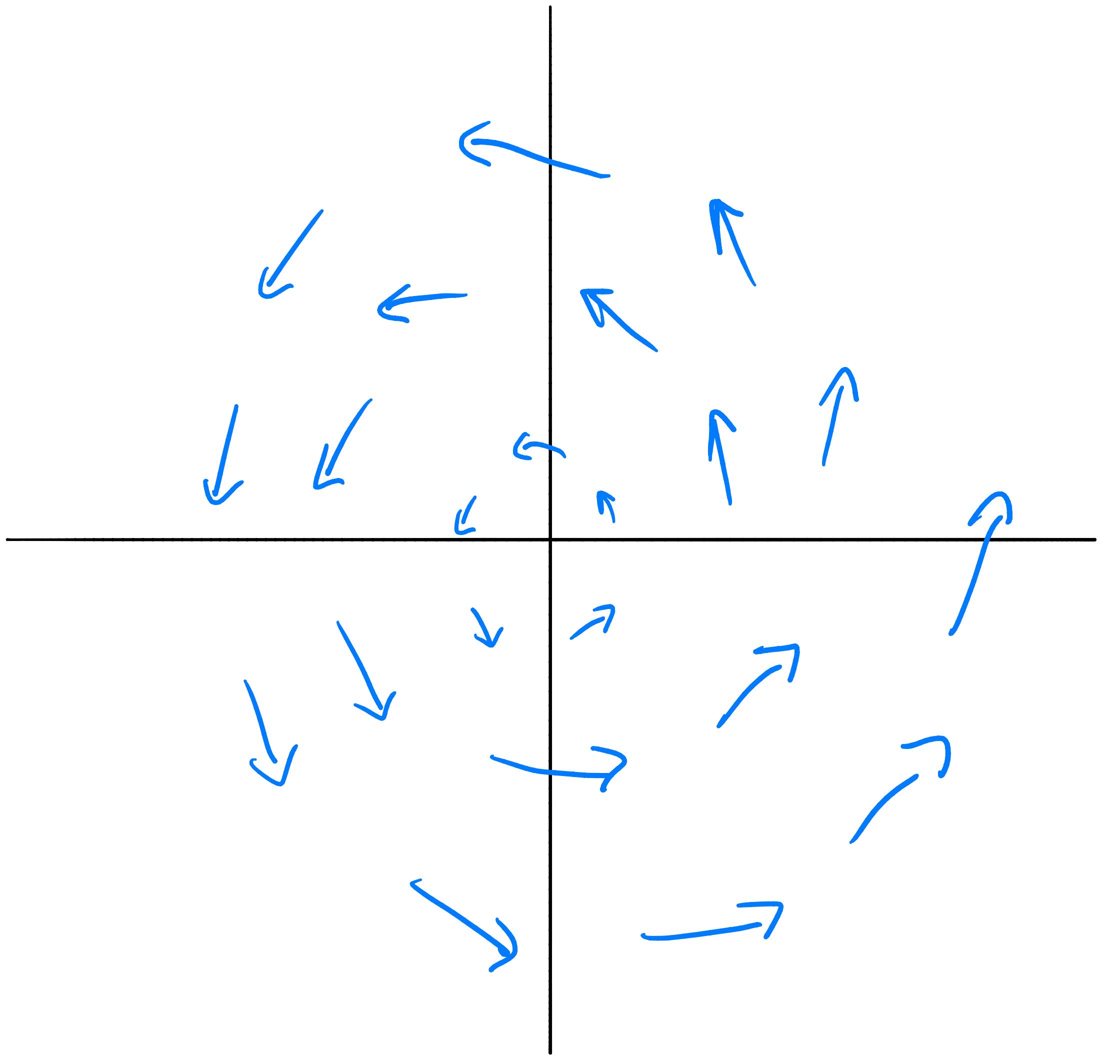
$$\gamma : [0,1] \to \mathbb{R}^2, \quad t \mapsto (tx, ty)$$

$$f(x,y) - f(0,0) = \int_\gamma F \, dl = \int_0^1 F(\gamma(t)) \cdot \dot{\gamma}(t) \, dt$$

$$= \int_0^1 (tx, ty) \cdot (x,y) \, dt = (x^2 + y^2) \int_0^1 t \, dt = \frac{1}{2} x^2 + \frac{1}{2} y^2$$

Hence $f(x,y) = \frac{1}{2} x^2 + \frac{1}{2} y^2 + C$

Example: $\quad F(x, y) = (-y, x)$



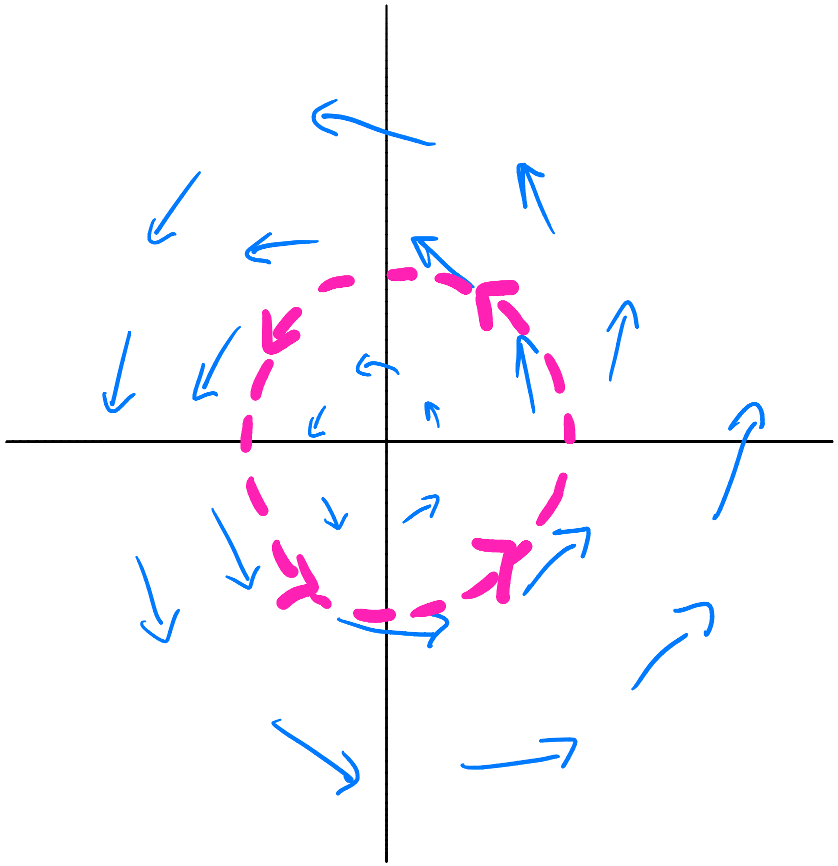
This vector field models a rotational flow.

Obviously

$$\operatorname{div} F = 0$$

$$\operatorname{curl} F = 2$$

We verify the theorems of Gauss and Green at a circular region



Consider the regular closed curve

$$\gamma : [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos t, \sin t)$$

Tangential vector

$$\dot{\gamma}(t) = (-\sin t, \cos t)$$

Unit tangent vector

$$T(t) = (-\sin t, \cos t)$$

Unit normal vector

$$n(t) = \frac{(\dot{\gamma}_2(t), -\dot{\gamma}_1(t))}{\|\dot{\gamma}(t)\|} = (\cos t, \sin t)$$

Verify divergence theorem

$$0 = \iint_{\Omega} \operatorname{div} F \, dx dy = \oint_{\partial\Omega} F \cdot \vec{n} \, |\dot{\gamma}(t)| \, dl$$

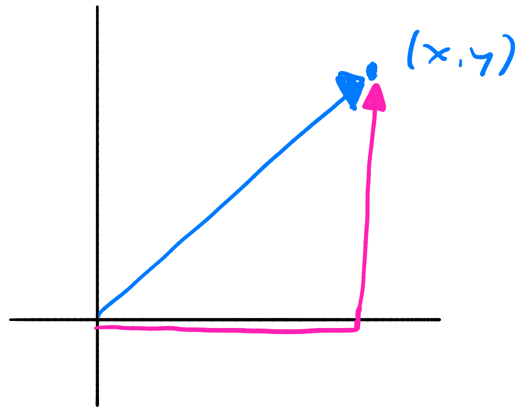
$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (\cos t, \sin t) \, dt = 0$$

Verify Green's theorem

$$\iint_{\Omega} \operatorname{curl} F \, dx dy = \int_{\Omega} 2 \, dx dy = \operatorname{area}(\Omega) \cdot 2 = 2\pi$$

$$\oint_{\partial\Omega} F \, dl = \int_0^{2\pi} F(\gamma(t)) \cdot \dot{\gamma}(t) \, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t) \, dt = 2\pi$$

What if we try to construct a potential?
Depending on the path, we may get different values for the potential

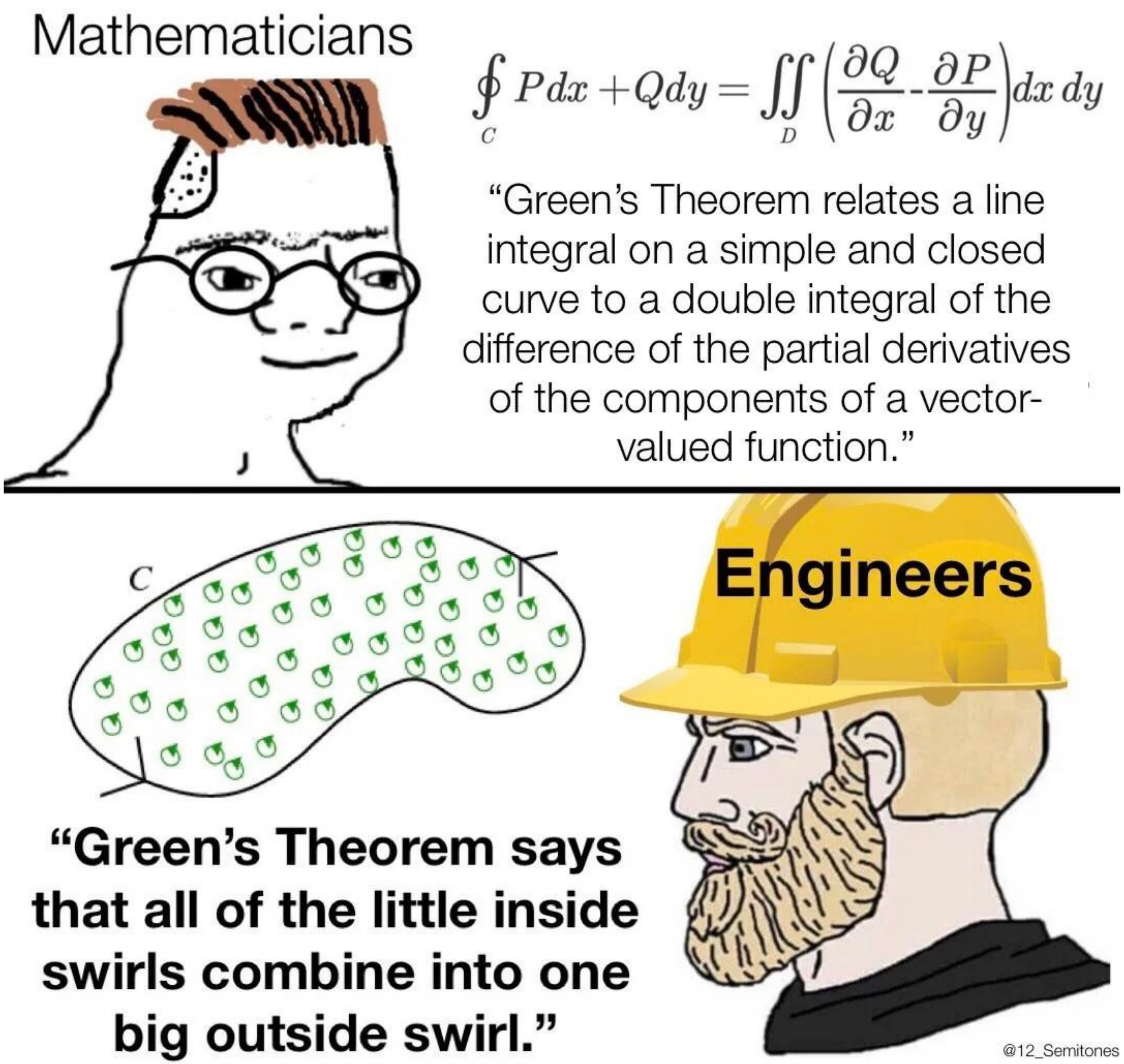


$$\gamma : [0,1] \to \mathbb{R}^2, \quad t \mapsto (tx, ty)$$

$$\int_\gamma F\, d\ell = \int_0^1 F(\gamma(t)) \cdot \dot{\gamma}(t)\, dt = \int_0^1 (-ty, tx) \cdot (x,y)\, dt = 0$$

$$\delta : [0,2] \to \mathbb{R}^2, \quad t \mapsto \begin{cases} (tx, 0) & 0 \leq t < 1 \\ (x, (t-1)y) & 1 \leq t \leq 2 \end{cases}$$

$$\int_\delta F\, d\ell = \int_0^1 (0, tx) \cdot (x, 0)\, dt + \int_1^2 (-(t-1)y, x) \cdot (0, y)\, dt = 0 + xy$$


Mathematicians
$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$
"Green's Theorem relates a line integral on a simple and closed curve to a double integral of the difference of the partial derivatives of the components of a vector-valued function."
C
Engineers
"Green's Theorem says that all of the little inside swirls combine into one big outside swirl."
@12_Semitones